

# Trigonometric Identities

## Solutions

For AQA Level 2 Certificate in Further Mathematics (8365)

---

Each question is shown in grey, followed by the working, one step per line. The final answer is in a box. *Marks* shows how the marks are usually given in the exam, and **Tip** points out a common mistake.

---

**1** (1 mark)

Circle the expression that is identical to  $1 - \cos^2 \theta$

Options:  $\sin \theta$ ,  $\sin^2 \theta$ ,  $(1 - \cos \theta)^2$ ,  $\tan^2 \theta$

Rearrange  $\sin^2 \theta + \cos^2 \theta = 1$ :

subtract  $\cos^2 \theta$  from both sides to get  $\sin^2 \theta = 1 - \cos^2 \theta$

**Answer:**  $\sin^2 \theta$

**Tip:**  $1 - \cos^2 \theta$  is  $\sin^2 \theta$ , not  $\sin \theta$ . The squares stay.

---

**2** (3 marks)

Simplify

(a)  $\tan \theta \cos \theta$

Use  $\tan \theta = \frac{\sin \theta}{\cos \theta}$

$\tan \theta \cos \theta = \frac{\sin \theta}{\cos \theta} \times \cos \theta$

The  $\cos \theta$  cancels:  $= \sin \theta$

**Answer:**  $\sin \theta$

(b)  $5 - 5 \cos^2 \theta$

Take out the common factor:  $5 - 5 \cos^2 \theta = 5(1 - \cos^2 \theta)$

From  $\sin^2 \theta + \cos^2 \theta = 1$ ,  $1 - \cos^2 \theta = \sin^2 \theta$

So  $5 - 5 \cos^2 \theta = 5 \sin^2 \theta$

**Answer:**  $5 \sin^2 \theta$

**Tip:** Factorise first:  $5 - 5 \cos^2 \theta = 5(1 - \cos^2 \theta)$ . It is not  $5 - 5 = 0$  or  $(5 - 5 \cos \theta)^2$ .

(c)  $\frac{\sin^2 \theta}{1 - \sin^2 \theta}$

From  $\sin^2 \theta + \cos^2 \theta = 1$ ,  $1 - \sin^2 \theta = \cos^2 \theta$

$$\frac{\sin^2 \theta}{\cos^2 \theta} = \left( \frac{\sin \theta}{\cos \theta} \right)^2 = \tan^2 \theta$$

**Answer:**  $\tan^2 \theta$

---

**3** (4 marks)

$A$  is an acute angle.

$$\sin A = \frac{2}{5}$$

(a) Work out the exact value of  $\cos A$ .

$$\cos^2 A = 1 - \sin^2 A \quad (\text{from } \sin^2 A + \cos^2 A = 1)$$

$$\cos^2 A = 1 - \left( \frac{2}{5} \right)^2 = 1 - \frac{4}{25} = \frac{21}{25}$$

$A$  is acute, so  $\cos A$  is positive.

$$\cos A = \sqrt{\frac{21}{25}} = \frac{\sqrt{21}}{5}$$

**Answer:**  $\cos A = \frac{\sqrt{21}}{5}$

**Tip:** Square the whole fraction:  $\left( \frac{2}{5} \right)^2 = \frac{4}{25}$ , not  $\frac{4}{5}$ . You can also draw a right-angled triangle with opposite side 2 and hypotenuse 5.

*Marks:* 1 mark for  $\cos^2 A = \frac{21}{25}$  (or  $1 - \frac{4}{25}$ ). 1 mark for  $\frac{\sqrt{21}}{5}$ .

- (b) Work out the exact value of  $\tan A$ .  
Give your answer with a rational denominator.

$$\tan A = \frac{\sin A}{\cos A} = \frac{2}{5} \div \frac{\sqrt{21}}{5}$$

$$= \frac{2}{5} \times \frac{5}{\sqrt{21}} = \frac{2}{\sqrt{21}}$$

Multiply top and bottom by  $\sqrt{21}$ :  $\frac{2\sqrt{21}}{21}$

$$\text{Answer: } \tan A = \frac{2\sqrt{21}}{21}$$

Marks: 1 mark for  $\frac{2}{\sqrt{21}}$ , 1 mark for  $\frac{2\sqrt{21}}{21}$ .

---

**4** (2 marks)

Write  $\cos^2 \theta - 4 \sin \theta + 2$  in the form  $a \sin^2 \theta + b \sin \theta + c$  where  $a$ ,  $b$  and  $c$  are integers.

$$\text{From } \sin^2 \theta + \cos^2 \theta = 1, \cos^2 \theta = 1 - \sin^2 \theta$$

$$\cos^2 \theta - 4 \sin \theta + 2 = (1 - \sin^2 \theta) - 4 \sin \theta + 2$$

$$= -\sin^2 \theta - 4 \sin \theta + 3$$

$$\text{Answer: } -\sin^2 \theta - 4 \sin \theta + 3 \quad (a = -1, b = -4, c = 3)$$

**Tip:** Use  $\cos^2 \theta = 1 - \sin^2 \theta$  and put it in brackets. A common error is  $\cos \theta = 1 - \sin \theta$ , which is not true.

Marks: 1 mark for replacing  $\cos^2 \theta$  with  $1 - \sin^2 \theta$ . 1 mark for the correct answer.

---

**5** (2 marks)

Show that  $\frac{1 - \cos^2 \theta}{\sin \theta \cos \theta} \equiv \tan \theta$

Start with the left-hand side:  $\frac{1 - \cos^2 \theta}{\sin \theta \cos \theta}$

Using  $\sin^2 \theta + \cos^2 \theta = 1$ ,  $1 - \cos^2 \theta = \sin^2 \theta$ :

$$= \frac{\sin^2 \theta}{\sin \theta \cos \theta}$$

Cancel a factor of  $\sin \theta$ :  $= \frac{\sin \theta}{\cos \theta}$

Using  $\tan \theta = \frac{\sin \theta}{\cos \theta}$ :  $= \tan \theta$ , which is the right-hand side.

**Answer:**  $\frac{1 - \cos^2 \theta}{\sin \theta \cos \theta} = \frac{\sin^2 \theta}{\sin \theta \cos \theta} = \frac{\sin \theta}{\cos \theta} = \tan \theta$

**Tip:** Work from one side to the other. Don't write the identity at the top and change both sides at once.

*Marks:* 1 mark for replacing  $1 - \cos^2 \theta$  with  $\sin^2 \theta$ . 1 mark for cancelling to  $\frac{\sin \theta}{\cos \theta}$  and stating it is  $\tan \theta$ .

---

**6** (3 marks)

Prove that  $\tan \theta + \frac{1}{\tan \theta} \equiv \frac{1}{\sin \theta \cos \theta}$

Start with the left-hand side and use  $\tan \theta = \frac{\sin \theta}{\cos \theta}$ :

$$\tan \theta + \frac{1}{\tan \theta} = \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta}$$

Common denominator  $\sin \theta \cos \theta$ :  $= \frac{\sin^2 \theta}{\sin \theta \cos \theta} + \frac{\cos^2 \theta}{\sin \theta \cos \theta}$

$$= \frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta \cos \theta}$$

Using  $\sin^2 \theta + \cos^2 \theta = 1$ :  $= \frac{1}{\sin \theta \cos \theta}$ , which is the right-hand side.

$$\text{Answer: LHS} = \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} = \frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta \cos \theta} = \frac{1}{\sin \theta \cos \theta} = \text{RHS}$$

**Tip:** Don't cross-multiply as if it were an equation: that scored 0 or 1 mark in the exam. Start with the more complicated side and say which identity you use at each step.

*Marks:* 1 mark for writing  $\tan \theta$  as  $\frac{\sin \theta}{\cos \theta}$  and  $\frac{1}{\tan \theta}$  as  $\frac{\cos \theta}{\sin \theta}$ . 1 mark for adding over a common denominator. 1 mark for using  $\sin^2 \theta + \cos^2 \theta = 1$  to complete the proof.

---

**7** (3 marks)

$\theta$  is an angle between  $180^\circ$  and  $270^\circ$

$$\tan \theta = 2\sqrt{6}$$

Work out the exact value of  $\cos \theta$ .

$$\tan \theta = \frac{\sin \theta}{\cos \theta}, \text{ so } \sin \theta = 2\sqrt{6} \cos \theta$$

$$\text{Substitute into } \sin^2 \theta + \cos^2 \theta = 1: (2\sqrt{6} \cos \theta)^2 + \cos^2 \theta = 1$$

$$24 \cos^2 \theta + \cos^2 \theta = 1$$

$$25 \cos^2 \theta = 1, \text{ so } \cos^2 \theta = \frac{1}{25} \text{ and } \cos \theta = \pm \frac{1}{5}$$

(Or: a right-angled triangle with opposite  $2\sqrt{6}$  and adjacent 1 has hypotenuse  $\sqrt{24+1} = 5$ , so the size of  $\cos \theta$  is  $\frac{1}{5}$ .)

Between  $180^\circ$  and  $270^\circ$  (3rd quadrant)  $\cos$  is negative, so  $\cos \theta = -\frac{1}{5}$

$$\text{Answer: } \cos \theta = -\frac{1}{5}$$

**Tip:**  $(2\sqrt{6})^2 = 4 \times 6 = 24$ . Then use the range: between  $180^\circ$  and  $270^\circ$  both  $\sin$  and  $\cos$  are negative (only  $\tan$  is positive), so the answer is  $-\frac{1}{5}$ , not  $\frac{1}{5}$ .

*Marks:* 1 mark for  $\sin \theta = 2\sqrt{6} \cos \theta$  substituted into  $\sin^2 \theta + \cos^2 \theta = 1$  (or a correct triangle). 1 mark for  $\cos^2 \theta = \frac{1}{25}$  (or hypotenuse 5). 1 mark for  $-\frac{1}{5}$ .

---

**8** (3 marks)

Show that  $\frac{\cos^2 \theta}{1 + \sin \theta} + 2 \sin \theta \equiv 1 + \sin \theta$

Start with the left-hand side. Using  $\sin^2 \theta + \cos^2 \theta = 1$ ,  $\cos^2 \theta = 1 - \sin^2 \theta$ :

$$\frac{\cos^2 \theta}{1 + \sin \theta} + 2 \sin \theta = \frac{1 - \sin^2 \theta}{1 + \sin \theta} + 2 \sin \theta$$

$1 - \sin^2 \theta$  is a difference of two squares:  $1 - \sin^2 \theta = (1 - \sin \theta)(1 + \sin \theta)$

$$= \frac{(1 - \sin \theta)(1 + \sin \theta)}{1 + \sin \theta} + 2 \sin \theta$$

Cancel  $(1 + \sin \theta)$ :  $= 1 - \sin \theta + 2 \sin \theta$

$= 1 + \sin \theta$ , which is the right-hand side.

**Answer:** LHS =  $\frac{(1 - \sin \theta)(1 + \sin \theta)}{1 + \sin \theta} + 2 \sin \theta = 1 - \sin \theta + 2 \sin \theta = 1 + \sin \theta = \text{RHS}$

**Tip:** Look for factorising, as in algebraic fractions:  $1 - \sin^2 \theta$  factorises just like

$$1 - x^2 = (1 - x)(1 + x).$$

*Marks:* 1 mark for  $\cos^2 \theta = 1 - \sin^2 \theta$ . 1 mark for factorising  $1 - \sin^2 \theta$  as  $(1 - \sin \theta)(1 + \sin \theta)$ . 1 mark for cancelling and completing the proof.

**9** (3 marks)

The equation  $6 \cos^2 x + \sin x = 5$  can be written in the form

$$(3 \sin x + a)(2 \sin x + b) = 0$$

where  $a$  and  $b$  are integers.

Work out the values of  $a$  and  $b$ .

You must show your working.

Use  $\cos^2 x = 1 - \sin^2 x$  (from  $\sin^2 x + \cos^2 x = 1$ ) to write everything in terms of  $\sin x$ :

$$6(1 - \sin^2 x) + \sin x = 5$$

$$6 - 6 \sin^2 x + \sin x - 5 = 0$$

$$-6 \sin^2 x + \sin x + 1 = 0$$

$$\text{Multiply by } -1: 6 \sin^2 x - \sin x - 1 = 0$$

Factorise (like  $6y^2 - y - 1 = (3y + 1)(2y - 1)$ ):  $(3 \sin x + 1)(2 \sin x - 1) = 0$

So  $a = 1$  and  $b = -1$ .

**Answer:**  $(3 \sin x + 1)(2 \sin x - 1) = 0$ , so  $a = 1, b = -1$

**Tip:** Put brackets round  $1 - \sin^2 x$  before multiplying by 6:  $6(1 - \sin^2 x) = 6 - 6 \sin^2 x$ , not  $6 - \sin^2 x$ . Expand your brackets at the end to check.

*Marks:* 1 mark for replacing  $\cos^2 x$  with  $(1 - \sin^2 x)$ . 1 mark for  $6 \sin^2 x - \sin x - 1 = 0$  (or the same with all signs changed). 1 mark for the correct factorisation.

---