

Trigonometric Graphs and Ratios of Any Angle

Solutions

For AQA Level 2 Certificate in Further Mathematics (8365)

Each question is shown in grey, followed by the working, one step per line. The final answer is in a box. *Marks* shows how the marks are usually given in the exam, and **Tip** points out a common mistake.

1 (1 mark)

Circle the value of x for which $\cos x = -1$ and $0^\circ \leq x \leq 360^\circ$

Options: 90° , 180° , 270° , 360°

The graph of $y = \cos x$ starts at $(0, 1)$, crosses the x -axis at 90° and reaches its lowest value, -1 , at 180° .

So $\cos 180^\circ = -1$

Answer: 180°

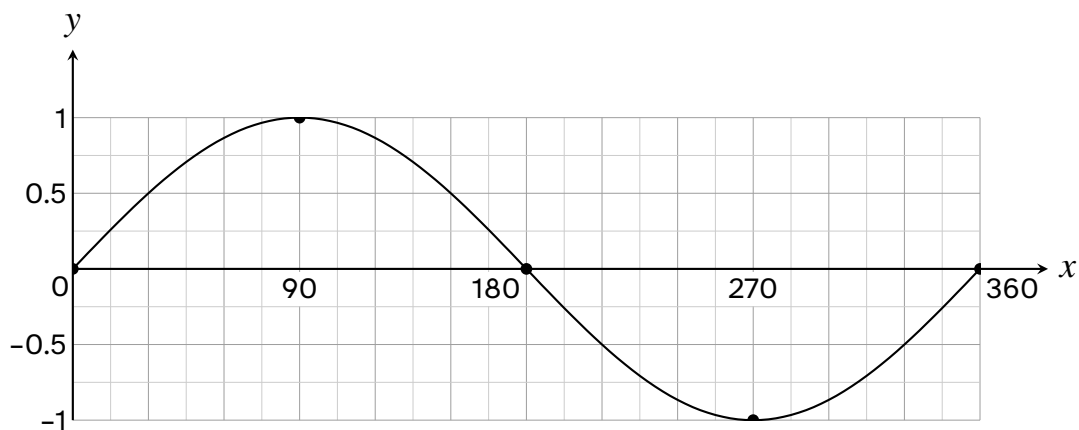
2 (2 marks)

On the axes, sketch $y = \sin x$ for $0^\circ \leq x \leq 360^\circ$

Key points: $\sin 0^\circ = 0$, $\sin 90^\circ = 1$, $\sin 180^\circ = 0$, $\sin 270^\circ = -1$, $\sin 360^\circ = 0$

Plot $(0, 0)$, $(90, 1)$, $(180, 0)$, $(270, -1)$ and $(360, 0)$.

Join them with one smooth wave: up from the origin, rounded at the top and bottom.

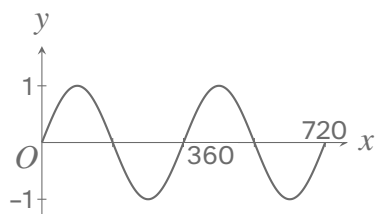


Tip: The curve must be rounded at $(90, 1)$ and $(270, -1)$. Examiners did not give the accuracy mark for sketches with a sharp point (a 'V') at a turning point or made from straight lines.

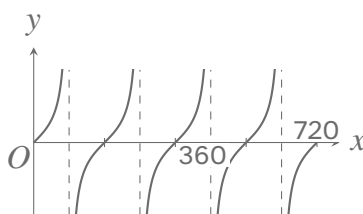
Marks: 1 mark for a wave that starts at the origin and goes up first. 1 mark for a smooth curve through all five key points.

3 (3 marks)

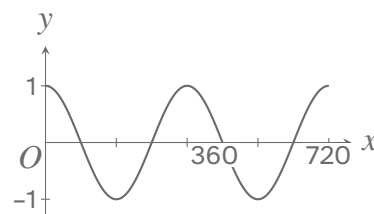
Here are sketches of $y = \sin x$, $y = \cos x$ and $y = \tan x$ for $0^\circ \leq x \leq 720^\circ$, in a different order.



A



B



C

- (a) Write down the letter of the graph of $y = \cos x$

$\cos 0^\circ = 1$, so the graph of $y = \cos x$ crosses the y-axis at 1.

Graph A goes through the origin ($y = \sin x$) and graph B has asymptotes ($y = \tan x$).

Graph C starts at 1.

Answer: C

- (b) Write down the number of solutions of $\tan x = 2$ for $0^\circ \leq x \leq 720^\circ$

Imagine the line $y = 2$ drawn across graph B.

$y = 2$ is above the x -axis, so the line only meets the parts of the curve above the x -axis.

The tan graph repeats every 180° : the line meets the curve once between 0° and 90° , once between 180° and 270° , once between 360° and 450° and once between 540° and 630° .

(Calculator: $x = 63.4^\circ$, then add 180° each time: 243.4° , 423.4° , 603.4° . The next one, 783.4° , is outside the range.)

Answer: 4

Tip: Take care at the ends of the range: the pieces of the tan graph next to 0° and 720° are only half-branches, and only the one starting at 0° goes above the x -axis.

- (c) Write down the number of solutions of $\cos x = 1$ for $0^\circ \leq x \leq 720^\circ$

$\cos x = 1$ at the top of each wave of graph C.

In this range that is at $x = 0$, $x = 360$ and $x = 720$.

Answer: 3

Tip: Check the ends of the range: 0° and 720° are included here (the sign is $\leq$), so they count.

4 (2 marks)

$$\sin 28^\circ = k$$

Write down, in terms of k , the value of

- (a) $\sin 152^\circ$

The graph of $y = \sin x$ is symmetrical about the line $x = 90$.

$152 = 180 - 28$, so 152° is the same distance from 180° as 28° is from 0° .

So $\sin 152^\circ = \sin 28^\circ = k$

Answer: k

(b) $\sin 332^\circ$

$$332 = 360 - 28$$

Between 180° and 360° the sine graph is below the x -axis, so $\sin 332^\circ$ is negative.

By the symmetry of the graph, it is the same size as $\sin 28^\circ$.

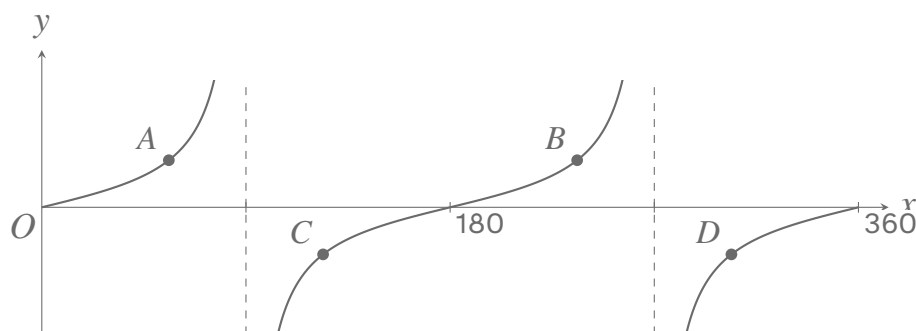
$$\text{So } \sin 332^\circ = -k$$

Answer: $-k$

Tip: Decide the sign first from a sketch of the graph (or CAST), then the size. Check on your calculator: $\sin 28^\circ = 0.469 \dots$ and $\sin 332^\circ = -0.469 \dots$

5 (3 marks)

Here is a sketch of $y = \tan x$ for $0^\circ \leq x \leq 360^\circ$



$A(56, p)$, B , C and D are points on the curve.

B has y -coordinate p

C and D have y -coordinate $-p$

(a) Write down the x -coordinate of B .

The graph of $y = \tan x$ repeats every 180° .

$$\text{So } \tan(56^\circ + 180^\circ) = \tan 56^\circ = p$$

$$56 + 180 = 236$$

Answer: 236

Tip: The tan graph repeats every 180° , not every 360° like sin and cos.

- (b) Work out the x -coordinates of C and D .

The tan graph has rotational symmetry about $(180, 0)$, so

$$\tan(180^\circ - 56^\circ) = -\tan 56^\circ = -p$$

$$C: 180 - 56 = 124$$

Add 180° for the next point with $y = -p$: $124 + 180 = 304$

$$D: 304 \text{ (this is also } 360 - 56)$$

Answer: $C: 124, D: 304$

Tip: tan is negative in the 2nd and 4th quadrants (between 90° and 180° , and between 270° and 360°), so the answers are $180 - 56$ and $360 - 56$.

Marks: 1 mark for each correct value.

6 (2 marks)

- (a) The equation $\sin x = k$ has exactly three solutions for $0^\circ \leq x \leq 360^\circ$
Write down the value of k .

Picture horizontal lines $y = k$ crossing the sine graph for $0^\circ \leq x \leq 360^\circ$.

For most values of k between -1 and 1 there are two crossings.

The line $y = 0$ (the x -axis) meets the curve at $x = 0, 180$ and 360 : three solutions.

Answer: $k = 0$

- (b) The equation $\sin x = m$ has exactly one solution for $0^\circ \leq x \leq 360^\circ$
Write down the two possible values of m .

For $-1 < m < 1$ (except $m = 0$) the line $y = m$ crosses the sine graph twice, and $m = 0$ gives three solutions.

The line $y = 1$ just touches the curve at its highest point, $x = 90$: one solution.

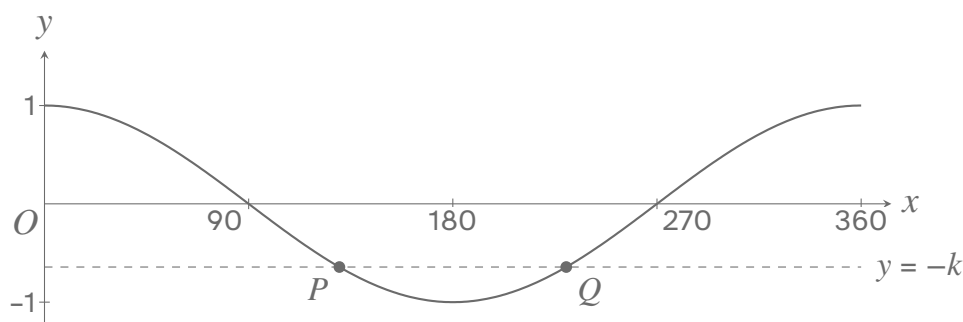
The line $y = -1$ just touches the curve at its lowest point, $x = 270$: one solution.

Answer: $m = 1$ or $m = -1$

Tip: Think about horizontal lines that only touch the curve: they are at the maximum and minimum points.

7 (3 marks)

Here is a sketch of $y = \cos x$ for $0^\circ \leq x \leq 360^\circ$



The line $y = -k$ meets the curve at the points $P(130, -k)$ and Q .

- (a) Write down the x -coordinate of Q .

The cosine graph is symmetrical about the line $x = 180$.

P is $180 - 130 = 50$ to the left of $x = 180$, so Q is 50 to the right.

$$180 + 50 = 230$$

Answer: 230

- (b) The line $y = k$ meets the curve at two points.
Work out the x -coordinates of these two points.

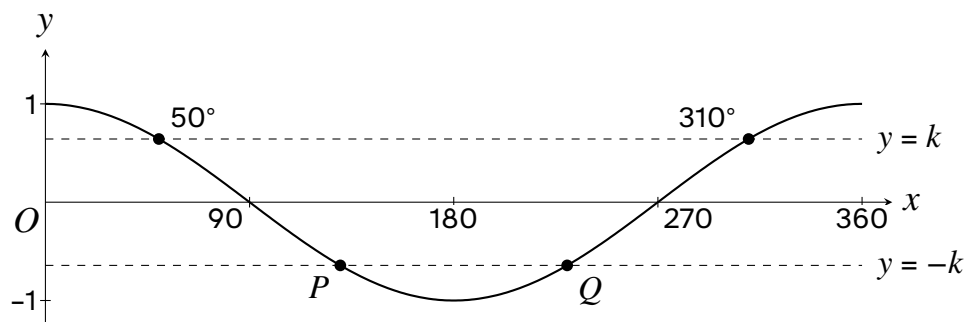
P is on the curve, so $\cos 130^\circ = -k$.

The graph has rotational symmetry about $(90, 0)$, so

$$\cos(180^\circ - 130^\circ) = -\cos 130^\circ = k$$

First point: $x = 180 - 130 = 50$

The graph is symmetrical about $x = 180$, so the second point is at
 $x = 360 - 50 = 310$



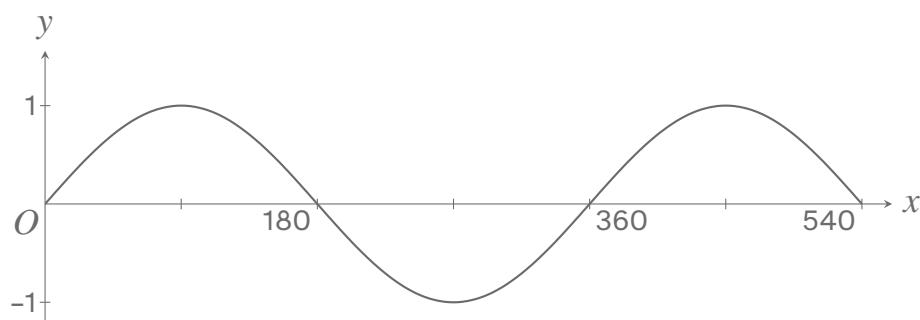
Answer: 50 and 310

Tip: Use the symmetry of the graph, not a calculator guess. A common slip is to give 230, which is where the curve meets $y = -k$, not $y = k$.

Marks: 1 mark for each correct value.

8 (4 marks)

Here is a sketch of $y = \sin x$ for $0^\circ \leq x \leq 540^\circ$



β is an acute angle.

$$\sin \beta = k$$

- (a) Write down, in terms of β , all the **other** values of x between 0° and 360° for which $\sin x = k$ or $\sin x = -k$

$\sin x = k$: the graph is symmetrical about the line $x = 90$, so the other value is the same distance from 180° as β is from 0° : $180^\circ - \beta$

$\sin x = -k$ is negative, so x is between 180° and 360° (the curve is below the x -axis there).

The graph has rotational symmetry about $(180, 0)$, so

$$\sin(180^\circ + \beta) = -\sin \beta = -k$$

The graph is symmetrical about $x = 270$, so the other value is

$$540^\circ - (180^\circ + \beta) = 360^\circ - \beta$$

Check with $\beta = 30^\circ$: $\sin 150^\circ = 0.5$ and $\sin 210^\circ = \sin 330^\circ = -0.5$ ✓

Answer: $\sin x = k$: $180^\circ - \beta$; $\sin x = -k$: $180^\circ + \beta$ and $360^\circ - \beta$

Tip: Mark β on a sketch first, then use the symmetry of the curve. Test your expressions with a real acute angle such as $\beta = 30^\circ$ on your calculator: a common slip is $90^\circ + \beta$ or $270^\circ - \beta$, which give $\pm \cos \beta$, not $\pm k$.

Marks: 1 mark for $180^\circ - \beta$. 1 mark for both $180^\circ + \beta$ and $360^\circ - \beta$.

- (b) Work out, in terms of β , the two values of x between 360° and 540° for which $\sin x = k$

The sine graph repeats every 360° .

From part (a), β and $180^\circ - \beta$ are the two values between 0° and 180° with $\sin x = k$.

Add 360° to each: $360^\circ + \beta$ and $360^\circ + 180^\circ - \beta = 540^\circ - \beta$

β is acute, so both are between 360° and 540° .

Answer: $360^\circ + \beta$ and $540^\circ - \beta$

Marks: 1 mark for each correct expression.

9 (2 marks)

θ is an angle between 0° and 360°

$\cos \theta = 0.4$ and $\tan \theta$ is negative.

Work out the value of θ .

Give your answer to 1 decimal place.

$\tan \theta = \frac{\sin \theta}{\cos \theta}$ is negative and $\cos \theta$ is positive, so $\sin \theta$ is negative.

$\cos$ positive and $\sin$ negative: θ is between 270° and 360° (the 4th quadrant).

Calculator: $\cos^{-1} 0.4 = 66.42 \dots^\circ$ (this is in the 1st quadrant, where $\tan$ is positive)

By the symmetry of the cosine graph about $x = 180$: $\theta = 360 - 66.42 \dots = 293.57 \dots$

$$\theta = 293.6^\circ \text{ (1 d.p.)}$$

Answer: $\theta = 293.6^\circ$

Tip: Give the angle in the range asked. 66.4° has $\tan \theta$ positive, so it is not the answer. Use CAST or the graphs to find the quadrant before you use the calculator.

Marks: 1 mark for 66.4° or for showing θ is between 270° and 360° . 1 mark for 293.6° .
