

Tangents and Normals

Solutions

For AQA Level 2 Certificate in Further Mathematics (8365)

Each question is shown in grey, followed by the working, one step per line. The final answer is in a box. *Marks* shows how the marks are usually given in the exam, and **Tip** points out a common mistake.

1 (1 mark)

The point $(-1, 2)$ lies on the curve $y = 2x^2$

Circle the gradient of the normal to the curve at $(-1, 2)$

Options: -4 , $-\frac{1}{4}$, $\frac{1}{4}$, 4

$$\frac{dy}{dx} = 4x$$

At $x = -1$: gradient of the curve (and of the tangent) $= 4 \times (-1) = -4$

The normal is perpendicular to the tangent, so its gradient is the negative reciprocal of -4 :

$$-\frac{1}{-4} = \frac{1}{4}$$

Answer: $\frac{1}{4}$

Tip: -4 is the gradient of the **tangent**. For the normal, flip the fraction and change the sign:
 $-4 \rightarrow \frac{1}{4}$.

2 (3 marks)

A curve has equation $y = x^3 - 2x^2 + 5$

The point $P(2, 5)$ lies on the curve.

Work out the equation of the tangent to the curve at P .

Give your answer in the form $y = mx + c$

$$\frac{dy}{dx} = 3x^2 - 4x$$

Gradient at P : substitute $x = 2$ into $\frac{dy}{dx}$: $3(2)^2 - 4(2) = 12 - 8 = 4$

The tangent has the same gradient as the curve at P , so $m = 4$.

Use $y - y_1 = m(x - x_1)$ with $(2, 5)$: $y - 5 = 4(x - 2)$

$$y - 5 = 4x - 8$$

$$y = 4x - 3$$

Answer: $y = 4x - 3$

Tip: Read the question: it asks for the **tangent**, so use the gradient 4 itself. Many students find the normal by mistake. Substitute $x = 2$ into $\frac{dy}{dx}$, not into y .

Marks: 1 mark for $\frac{dy}{dx} = 3x^2 - 4x$. 1 mark for the gradient 4. 1 mark for $y = 4x - 3$.

3 (4 marks)

A curve has equation $y = x^2 - \frac{8}{x}$

P is the point on the curve where $x = 2$

Show that the normal to the curve at P has equation $x + 6y = 2$

Find the y -coordinate of P : $y = 2^2 - \frac{8}{2} = 4 - 4 = 0$, so P is $(2, 0)$.

Write $y = x^2 - 8x^{-1}$, so $\frac{dy}{dx} = 2x + 8x^{-2}$

Gradient of the curve at P : $2(2) + \frac{8}{2^2} = 4 + 2 = 6$

The normal is perpendicular to the tangent, so its gradient is $-\frac{1}{6}$

Equation of the normal through $(2, 0)$: $y - 0 = -\frac{1}{6}(x - 2)$

Multiply both sides by 6: $6y = -(x - 2) = -x + 2$

Add x to both sides: $x + 6y = 2$ as required.

Answer: The normal has gradient $-\frac{1}{6}$ and passes through $(2, 0)$, giving $x + 6y = 2$

Tip: $-\frac{8}{x} = -8x^{-1}$ differentiates to $+8x^{-2}$. In a “show that”, write every step, including finding the y -coordinate of P , and finish at the given equation.

Marks: 1 mark for $\frac{dy}{dx} = 2x + 8x^{-2}$. 1 mark for the gradient 6 at P . 1 mark for the normal gradient $-\frac{1}{6}$ with the point $(2, 0)$. 1 mark for correct working leading to $x + 6y = 2$.

4 (3 marks)

A curve has equation $y = x^3 - 3x^2 - 7x + 2$

The tangents to the curve at the points A and B are both parallel to the line $y = 2x - 5$

Work out the x -coordinates of A and B .

Parallel lines have the same gradient, so the tangents at A and B have gradient 2.

$$\frac{dy}{dx} = 3x^2 - 6x - 7$$

Put the gradient equal to 2: $3x^2 - 6x - 7 = 2$

$$3x^2 - 6x - 9 = 0$$

Divide by 3: $x^2 - 2x - 3 = 0$

$$(x - 3)(x + 1) = 0$$

$$x = 3 \text{ or } x = -1$$

Answer: $x = -1$ and $x = 3$

Tip: The gradient of $y = 2x - 5$ is 2, the number in front of x . Put $\frac{dy}{dx}$ equal to 2, not y .

Marks: 1 mark for $\frac{dy}{dx} = 3x^2 - 6x - 7$. 1 mark for putting it equal to 2. 1 mark for both values.

5 (3 marks)

A curve has equation $y = kx^3 - 6x$ where k is a constant.

The normal to the curve at the point where $x = -1$ is parallel to the line $x + 3y = 4$

Work out the value of k .

Rearrange the line: $3y = -x + 4$, so $y = -\frac{1}{3}x + \frac{4}{3}$. Its gradient is $-\frac{1}{3}$

The normal is parallel to this line, so the gradient of the normal is $-\frac{1}{3}$

The tangent is perpendicular to the normal, so the gradient of the curve at $x = -1$ is 3 (the negative reciprocal of $-\frac{1}{3}$).

$$\frac{dy}{dx} = 3kx^2 - 6$$

$$\text{At } x = -1: 3k(-1)^2 - 6 = 3$$

$$3k - 6 = 3$$

$$3k = 9, \text{ so } k = 3$$

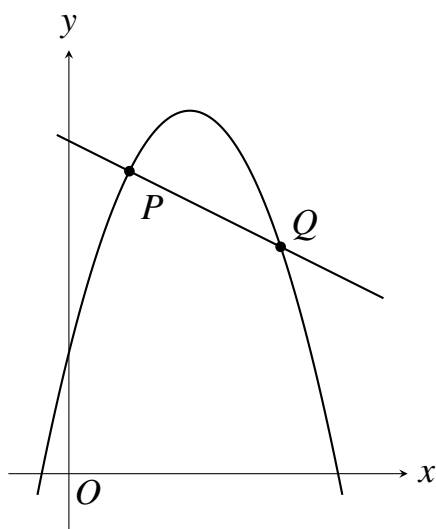
Answer: $k = 3$

Tip: Rearrange to $y = \dots$ before reading off a gradient: the gradient of $x + 3y = 4$ is $-\frac{1}{3}$, not 1 or 3. Then remember that $-\frac{1}{3}$ is the gradient of the **normal**; the curve's gradient is its negative reciprocal.

Marks: 1 mark for the gradient of the curve at $x = -1$ being 3. 1 mark for $3k(-1)^2 - 6 = 3$. 1 mark for $k = 3$.

6 (5 marks)

Here is a sketch of the curve $y = 2 + 4x - x^2$



Not drawn accurately

The point $P(1, 5)$ lies on the curve.

The normal to the curve at P meets the curve again at Q .

Work out the coordinates of Q .

You must show your working.

$$\frac{dy}{dx} = 4 - 2x$$

Gradient of the curve at P : $4 - 2(1) = 2$

The normal is perpendicular to the tangent, so its gradient is $-\frac{1}{2}$

Normal through $P(1, 5)$: $y - 5 = -\frac{1}{2}(x - 1)$

Multiply by 2: $2y - 10 = -x + 1$, so $x + 2y = 11$, or $y = \frac{11 - x}{2}$

Where the normal meets the curve, the y values are equal: $2 + 4x - x^2 = \frac{11 - x}{2}$

Multiply by 2: $4 + 8x - 2x^2 = 11 - x$

Rearrange: $2x^2 - 9x + 7 = 0$

The normal passes through P , so $x = 1$ is one solution and $(x - 1)$ is a factor.

$$(x - 1)(2x - 7) = 0$$

$x = 1$ is the point P , so Q has $x = \frac{7}{2}$

$$y = \frac{11 - 3.5}{2} = \frac{7.5}{2} = 3.75$$

Check on the curve: $2 + 4(3.5) - 3.5^2 = 2 + 14 - 12.25 = 3.75$

Answer: Q is $\left(\frac{7}{2}, \frac{15}{4}\right)$ (or $(3.5, 3.75)$)

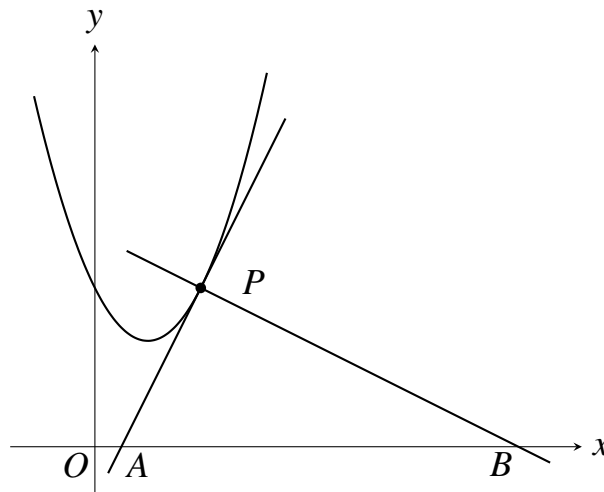
Tip: The normal uses the negative reciprocal of 2, which is $-\frac{1}{2}$. Using 2 itself gives the tangent, a common error. Substitute the line into the curve, and remember $x = 1$ is P , so $(x - 1)$ is a factor. Do not forget the y -coordinate of Q .

Marks: 1 mark for the gradient 2 at P . 1 mark for the normal gradient $-\frac{1}{2}$. 1 mark for the equation of the normal. 1 mark for substituting to get $2x^2 - 9x + 7 = 0$. 1 mark for $(3.5, 3.75)$.

7 (5 marks)

Here is a sketch of the curve $y = x^2 - 2x + 3$

P is the point on the curve where $x = 2$



Not drawn accurately

The tangent to the curve at P meets the x -axis at A .

The normal to the curve at P meets the x -axis at B .

Work out the area of triangle PAB .

$$\frac{dy}{dx} = 2x - 2, \text{ so the gradient at } P \text{ is } 2(2) - 2 = 2$$

Tangent: $y - 3 = 2(x - 2)$, so $y = 2x - 1$

At A , $y = 0$: $2x - 1 = 0$, so A is $(0.5, 0)$

The normal has gradient $-\frac{1}{2}$ (the negative reciprocal of 2).

Normal: $y - 3 = -\frac{1}{2}(x - 2)$, so $y = -\frac{1}{2}x + 4$

At B , $y = 0$: $\frac{1}{2}x = 4$, so B is $(8, 0)$

Base $AB = 8 - 0.5 = 7.5$

Height = the y -coordinate of $P = 3$

Area = $\frac{1}{2} \times 7.5 \times 3 = 11.25$

Answer: 11.25 (or $\frac{45}{4}$)

Tip: The tangent uses the gradient 2; the normal uses $-\frac{1}{2}$. Mixing them up was a common error in the exam. The height of the triangle is the y -coordinate of P , and the area is $\frac{1}{2} \times \text{base} \times \text{height}$.

Marks: 1 mark for the gradient 2 at P . 1 mark for $A(0.5, 0)$. 1 mark for the normal gradient $-\frac{1}{2}$ used in an equation through P . 1 mark for $B(8, 0)$. 1 mark for 11.25.
