

Stationary Points, Increasing and Decreasing Functions

Solutions

For AQA Level 2 Certificate in Further Mathematics (8365)

Each question is shown in grey, followed by the working, one step per line. The final answer is in a box. *Marks* shows how the marks are usually given in the exam, and **Tip** points out a common mistake.

1 (4 marks)

A curve has equation $y = x^2 + \frac{16}{x}$

(a) Show that the curve has a stationary point when $x = 2$

Write the fraction as a power of x : $y = x^2 + 16x^{-1}$

$$\frac{dy}{dx} = 2x - 16x^{-2} = 2x - \frac{16}{x^2}$$

$$\text{When } x = 2: \frac{dy}{dx} = 2 \times 2 - \frac{16}{2^2} = 4 - 4 = 0$$

$$\frac{dy}{dx} = 0 \text{ when } x = 2, \text{ so there is a stationary point at } x = 2$$

Answer: $\frac{dy}{dx} = 2x - \frac{16}{x^2} = 0$ when $x = 2$, so the curve has a stationary point there

Tip: Write $\frac{16}{x}$ as $16x^{-1}$ before you differentiate. The power goes **down** by 1, from -1 to -2 .

Marks: 1 mark for $2x - 16x^{-2}$. 1 mark for substituting $x = 2$ to get 0, with a conclusion.

(b) Use $\frac{d^2y}{dx^2}$ to determine the nature of this stationary point.
You must show your working.

$$\frac{dy}{dx} = 2x - 16x^{-2}$$

$$\frac{d^2y}{dx^2} = 2 + 32x^{-3} = 2 + \frac{32}{x^3}$$

When $x = 2$: $\frac{d^2y}{dx^2} = 2 + \frac{32}{8} = 6$

$\frac{d^2y}{dx^2} > 0$, so the stationary point is a minimum.

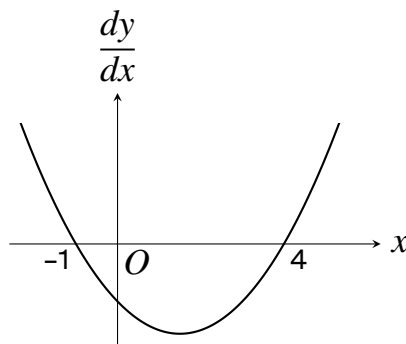
Answer: Minimum, because $\frac{d^2y}{dx^2} = 6 > 0$ when $x = 2$

Tip: $-16x^{-2}$ differentiates to $+32x^{-3}$: multiply by -2 , then subtract 1 from the power. The two minus signs make a plus. Positive second derivative means minimum.

Marks: 1 mark for $2 + 32x^{-3}$. 1 mark for 6 (or a positive value) and minimum.

2 (3 marks)

The sketch shows the graph of the gradient function $\frac{dy}{dx}$ for a curve $y = f(x)$



Not drawn accurately

- (a) Write down the x -coordinates of the stationary points of the curve $y = f(x)$

At a stationary point the gradient is zero, so $\frac{dy}{dx} = 0$

The graph of $\frac{dy}{dx}$ crosses the x -axis at $x = -1$ and $x = 4$

Answer: $x = -1$ and $x = 4$

Tip: Read where the graph of $\frac{dy}{dx}$ is **zero** (on the x -axis). The lowest point of this graph is not a stationary point of $y = f(x)$.

- (b) Which of these describes the curve $y = f(x)$ for $-1 < x < 4$?

Circle your answer.

Options: increasing, decreasing, increasing then decreasing, decreasing then increasing

For $-1 < x < 4$ the graph of $\frac{dy}{dx}$ is below the x -axis.

So the gradient of $y = f(x)$ is negative for all of these values.

A negative gradient means the curve is decreasing.

Answer: decreasing

Tip: Don't describe the shape of the $\frac{dy}{dx}$ graph. Look at its **sign**: below the axis means $y = f(x)$ is decreasing, above the axis means increasing.

- (c) Is the stationary point at $x = 4$ a maximum or a minimum? Tick one box.
Give a reason for your answer.

Just to the left of $x = 4$, $\frac{dy}{dx} < 0$, so the curve is going down.

Just to the right of $x = 4$, $\frac{dy}{dx} > 0$, so the curve is going up.

Down then up means a minimum.

Answer: Minimum: the gradient changes from negative to positive at $x = 4$

3 (5 marks)

The curve $y = 2x^3 + 9x^2 - 4$ has two stationary points.

Find both stationary points and determine the nature of each one.

You must show your working.

$$\frac{dy}{dx} = 6x^2 + 18x$$

$$\text{At a stationary point } \frac{dy}{dx} = 0: 6x^2 + 18x = 0$$

$$\text{Take out the common factor: } 6x(x + 3) = 0$$

$$\text{So } x = 0 \text{ or } x = -3$$

When $x = 0$: $y = 0 + 0 - 4 = -4$

When $x = -3$: $y = 2(-27) + 9(9) - 4 = -54 + 81 - 4 = 23$

$$\frac{d^2y}{dx^2} = 12x + 18$$

At $x = 0$: $\frac{d^2y}{dx^2} = 18 > 0$, so $(0, -4)$ is a minimum.

At $x = -3$: $\frac{d^2y}{dx^2} = -36 + 18 = -18 < 0$, so $(-3, 23)$ is a maximum.

Answer: $(0, -4)$ minimum; $(-3, 23)$ maximum

Tip: Factorise by taking out $6x$. If you divide by x you lose the solution $x = 0$. You must **test** the nature: just writing max and min, or drawing a sketch, is not a test.

Marks: 1 mark for $6x^2 + 18x$. 1 mark for $x = 0$ and $x = -3$. 1 mark for $y = -4$ and $y = 23$. 1 mark for a correct test of the nature (the second derivative, or the gradient either side). 1 mark for both natures correct.

4 (3 marks)

$$f(x) = x^3 - 6x^2 - 15x + 2$$

For which values of x is $f(x)$ decreasing?

Give your answer as an inequality.

$$f'(x) = 3x^2 - 12x - 15$$

$f(x)$ is decreasing when the gradient is negative: $3x^2 - 12x - 15 < 0$

Divide by 3: $x^2 - 4x - 5 < 0$

Factorise: $(x - 5)(x + 1) < 0$

The graph of $(x - 5)(x + 1)$ is a U-shape crossing the x -axis at $x = -1$ and $x = 5$

It is below the x -axis between these values.

Answer: $-1 < x < 5$

Tip: Decreasing means $f'(x) < 0$. For a U-shaped quadratic, that is **between** the roots. $x < -1$ or $x > 5$ is where $f(x)$ is increasing.

Marks: 1 mark for $3x^2 - 12x - 15$. 1 mark for the values -1 and 5 . 1 mark for $-1 < x < 5$.

5 (2 marks)

f is a cubic function.

The curve $y = f(x)$ has a maximum point at $(-2, 6)$ and a minimum point at $(3, -4)$

Tick one box for each statement.

	True	False
$\frac{dy}{dx} < 0$ for $-2 < x < 3$	✓	
The curve crosses the x -axis only once		✓
The tangent at $(3, -4)$ is the line $y = -4$	✓	
$\frac{d^2y}{dx^2} > 0$ when $x = -2$		✓

A cubic with a maximum on the left and a minimum on the right goes up, down, then up.

Between the maximum ($x = -2$) and the minimum ($x = 3$) the curve goes down, so $\frac{dy}{dx} < 0$. **True**

The maximum is above the x -axis and the minimum is below it, so the curve crosses the x -axis three times. **False**

The tangent at a turning point is horizontal, so at $(3, -4)$ it is the line $y = -4$. **True**

$x = -2$ is a maximum, so $\frac{d^2y}{dx^2}$ is negative (or zero) there, not positive. **False**

Answer: True, False, True, False

Tip: Draw a quick sketch with the two turning points marked. The tangent at any maximum or minimum point is parallel to the x -axis.

Marks: 2 marks for all four correct. 1 mark for three correct.

6 (3 marks)

f is a cubic function.

The curve $y = f(x)$

has a maximum point at $P(-3, 32)$

has a minimum point at $Q(1, 0)$

crosses the y -axis at $R(0, 5)$

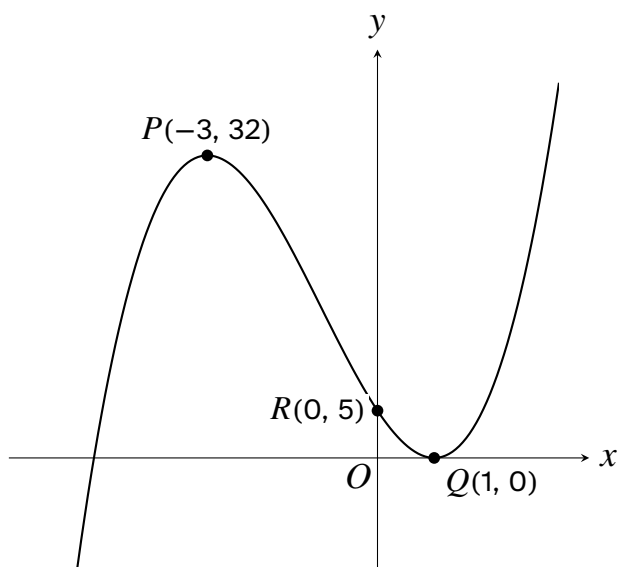
On the axes, sketch the curve. Label P , Q and R on your sketch.

The maximum P is to the left of the minimum Q , so the curve goes up, then down, then up (a positive cubic).

From $P(-3, 32)$ the curve goes down, through $R(0, 5)$ on the y -axis, to $Q(1, 0)$.

Q is a minimum point on the x -axis, so the curve **touches** the x -axis at Q and turns back up. It does not cross there.

To the left of P the curve goes down and crosses the x -axis once.



(The curve is $y = x^3 + 3x^2 - 9x + 5$, which crosses the x -axis at $x = -5$. You do not need this for the sketch.)

Answer: A cubic curve: up to a maximum at P , down through R on the y -axis, touching the x -axis at the minimum Q , then up again

Tip: A minimum point on the x -axis means the curve just touches the axis there. Draw one smooth curve, rounded at P and Q , not pointed.

Marks: 1 mark for a cubic shape with the maximum on the left and the minimum on the right. 1 mark for the minimum touching the x -axis at Q . 1 mark for P , Q and R labelled in the correct places.

7 (4 marks)

$$f(x) = x^3 - 9x^2 + 30x + 7$$

Show that $f(x)$ is increasing for every value of x .

$$f'(x) = 3x^2 - 18x + 30$$

Take out the factor 3: $f'(x) = 3(x^2 - 6x + 10)$

Complete the square: $x^2 - 6x + 10 = (x - 3)^2 - 9 + 10 = (x - 3)^2 + 1$

$$\text{So } f'(x) = 3(x - 3)^2 + 3$$

$(x - 3)^2 \geq 0$ for every value of x , so $f'(x) \geq 3$

So $f'(x) > 0$ for all x : the gradient is always positive.

Therefore $f(x)$ is an increasing function for all values of x .

Answer: $f'(x) = 3(x - 3)^2 + 3 \geq 3 > 0$ for all x , so $f(x)$ is increasing for all x

Tip: Use the **first** derivative, not the second. Substituting one or two values of x does not show it is positive for **all** x . Finish with a clear conclusion.

Marks: 1 mark for $3x^2 - 18x + 30$. 1 mark for $3(x - 3)^2 + 3$ (or $(x - 3)^2 + 1$). 1 mark for stating that $(x - 3)^2 \geq 0$, so $f'(x) > 0$. 1 mark for a full conclusion. (Showing the discriminant of $f'(x)$ is negative, $324 - 360 < 0$, with $f'(x) > 0$ somewhere, also works.)

8 (4 marks)

$g(x) = kx^3 - 6x^2 + 3x$, where k is a positive constant.

$g(x)$ is increasing for every value of x .

Work out the range of values of k .

Give your answer as an inequality.

$$g'(x) = 3kx^2 - 12x + 3$$

$g(x)$ is increasing for all x , so $g'(x) > 0$ for all x

$k > 0$, so $g'(x)$ is a U-shaped quadratic. It is always positive if it has no real roots.

No real roots means the discriminant is negative: $b^2 - 4ac < 0$

$$(-12)^2 - 4 \times 3k \times 3 < 0$$

$$144 - 36k < 0$$

$$144 < 36k$$

$$k > 4$$

Answer: $k > 4$

Tip: Your answer must not contain x . Completing the square also works:

$g'(x) = 3k\left(x - \frac{2}{k}\right)^2 + 3 - \frac{12}{k}$, which is always positive when $3 - \frac{12}{k} > 0$. (When $k = 4$,

$g'(x) = 3(2x - 1)^2$ is zero only at $x = \frac{1}{2}$, so $k \geq 4$ is often accepted too.)

Marks: 1 mark for $3kx^2 - 12x + 3$. 1 mark for using the discriminant (or completing the square).

1 mark for $144 - 36k < 0$ (or $3 - \frac{12}{k} > 0$). 1 mark for $k > 4$.

9 (6 marks)

The point $(2, -7)$ is a stationary point on the curve $y = x^3 + px^2 + qx + 1$

(a) Work out the values of p and q .

The point $(2, -7)$ is on the curve: $2^3 + p(2^2) + 2q + 1 = -7$

$$8 + 4p + 2q + 1 = -7, \text{ so } 4p + 2q = -16, \text{ so } 2p + q = -8 \quad (1)$$

$$\frac{dy}{dx} = 3x^2 + 2px + q$$

At a stationary point $\frac{dy}{dx} = 0$. When $x = 2$: $12 + 4p + q = 0$, so $4p + q = -12 \quad (2)$

$$(2) - (1): 2p = -4, \text{ so } p = -2$$

$$\text{Substitute into (1): } -4 + q = -8, \text{ so } q = -4$$

$$\textbf{Answer: } p = -2, q = -4$$

Tip: 'Stationary point' means $\frac{dy}{dx} = 0$. You need **two** equations: one from the point lying on the curve, and one from $\frac{dy}{dx} = 0$ when $x = 2$.

Marks: 1 mark for an equation from the point being on the curve. 1 mark for $3x^2 + 2px + q$. 1 mark for an equation from $\frac{dy}{dx} = 0$ at $x = 2$. 1 mark for both values.

(b) Work out the x -coordinate of the other stationary point of the curve.

$$\frac{dy}{dx} = 3x^2 - 4x - 4$$

$$\text{Set it equal to zero: } 3x^2 - 4x - 4 = 0$$

$$x = 2 \text{ is one solution, so } (x - 2) \text{ is a factor: } (3x + 2)(x - 2) = 0$$

$$\text{The other solution is } 3x + 2 = 0, \text{ so } x = -\frac{2}{3}$$

$$\textbf{Answer: } x = -\frac{2}{3}$$

Tip: You already know $x = 2$ is a solution, which helps you factorise.

Marks: 1 mark for $3x^2 - 4x - 4 = 0$ with your p and q . 1 mark for $-\frac{2}{3}$.