

Sequences - nth Terms and Limiting Values

Solutions

For AQA Level 2 Certificate in Further Mathematics (8365)

Each question is shown in grey, followed by the working, one step per line. The final answer is in a box. *Marks* shows how the marks are usually given in the exam, and **Tip** points out a common mistake.

1 (2 marks)

Here are the first four terms of a linear sequence.

13.6 11.2 8.8 6.4

Work out an expression for the n th term.

Common difference: $11.2 - 13.6 = -2.4$

So the n th term starts $-2.4n$

When $n = 1$, $-2.4n = -2.4$. To get to 13.6, add 16.

n th term = $16 - 2.4n$

Check $n = 4$: $16 - 9.6 = 6.4$ ✓

Answer: $16 - 2.4n$ (or $-2.4n + 16$)

Tip: The sequence is going down, so the difference is negative. The number with n is the difference, -2.4 , and the constant is the first term minus the difference: $13.6 - (-2.4) = 16$.

Marks: 1 mark for $-2.4n$. 1 mark for a fully correct expression.

2 (4 marks)

Here are the first four terms of a linear sequence.

247 240 233 226

- (a) Work out which term of the sequence has the value -145
You must show your working.

Common difference = -7

n th term = $254 - 7n$ (since $247 + 7 = 254$)

Solve $254 - 7n = -145$

$7n = 254 + 145 = 399$

$n = 57$

Answer: The 57th term

Marks: 1 mark for the n th term $254 - 7n$. 1 mark for 57.

- (b) How many terms of the sequence are positive?

The terms are positive when $254 - 7n > 0$

$7n < 254$

$n < 36.28 \dots$

n is a whole number, so $n = 1, 2, \dots, 36$

Check: 36th term = $254 - 252 = 2$ (positive) and 37th term = $254 - 259 = -5$ (negative) ✓

Answer: 36

Tip: n is a position in the sequence, so it must be a whole number. Check the terms either side of your boundary to be sure.

Marks: 1 mark for $254 - 7n > 0$ (or $= 0$), or $n = 36.28 \dots$. 1 mark for 36.

3 (5 marks)

Here are the first three terms of a linear sequence.

$$5p + 2q \qquad 7p - q \qquad 9p - 4q$$

- (a) Work out an expression for the n th term.
Give your answer in terms of n , p and q .

$$\text{Common difference: } (7p - q) - (5p + 2q) = 2p - 3q$$

n th term = first term $+(n - 1) \times$ difference

$$= 5p + 2q + (n - 1)(2p - 3q)$$

$$= 5p + 2q + 2pn - 3qn - 2p + 3q$$

$$= 2pn + 3p - 3qn + 5q$$

$$= (2n + 3)p + (5 - 3n)q$$

$$\text{Check } n = 3: 9p - 4q \quad \checkmark$$

Answer: $(2n + 3)p + (5 - 3n)q$ (or $2pn - 3qn + 3p + 5q$)

Tip: Use brackets when subtracting terms: $(7p - q) - (5p + 2q) = 2p - 3q$. Check your expression gives the 3rd term when $n = 3$.

Marks: 1 mark for the difference $2p - 3q$ or for $5p + 2q + (n - 1)(2p - 3q)$. 1 mark for a fully correct simplified expression.

- (b) The 1st term of the sequence is 31 and the 20th term is 50
Work out the values of p and q .

$$\text{1st term: } 5p + 2q = 31$$

From the 1st term to the 20th term there are 19 differences: $50 - 31 = 19$, so the difference is 1.

$$\text{So } 2p - 3q = 1$$

$$\text{Multiply the first equation by 3: } 15p + 6q = 93$$

$$\text{Multiply the second equation by 2: } 4p - 6q = 2$$

$$\text{Add: } 19p = 95, \text{ so } p = 5$$

$$\text{Substitute: } 25 + 2q = 31, \text{ so } q = 3$$

(Or use the 20th term from part (a): $43p - 55q = 50$ with $5p + 2q = 31$.)

Check: the terms are 31, 32, 33,... and the 20th term is $31 + 19 = 50$ ✓

Answer: $p = 5, q = 3$

Tip: The 20th term is the 1st term plus 19 differences, not 20.

Marks: 1 mark for $5p + 2q = 31$ with $2p - 3q = 1$ (or $43p - 55q = 50$). 1 mark for a correct method to solve the two equations. 1 mark for both values.

4 (5 marks)

Here are the first five terms of a quadratic sequence.

9 13 13 9 1

(a) Work out an expression for the n th term.

First differences: 4, 0, -4, -8

Second difference: -4, so the n^2 term is $\frac{-4}{2}n^2 = -2n^2$

Work out the sequence minus $-2n^2$ (that is, add $2n^2$ to each term):

$9 + 2 = 11, \quad 13 + 8 = 21, \quad 13 + 18 = 31, \quad 9 + 32 = 41$

11, 21, 31, 41 has n th term $10n + 1$

n th term = $-2n^2 + 10n + 1$

Check $n = 5$: $-50 + 50 + 1 = 1$ ✓

Answer: $-2n^2 + 10n + 1$

Tip: Half the second difference gives the n^2 coefficient. Here the second difference is negative, so the coefficient is -2. Check your expression with the first few terms.

Marks: 1 mark for $-2n^2$. 1 mark for $10n$ or $+1$ (from a correct method). 1 mark for $-2n^2 + 10n + 1$.

(b) Which term of the sequence has the value -131?
You must show your working.

Solve $-2n^2 + 10n + 1 = -131$

$$-2n^2 + 10n + 132 = 0$$

Divide by -2 : $n^2 - 5n - 66 = 0$

Factorise: $(n - 11)(n + 6) = 0$

$$n = 11 \text{ or } n = -6$$

n is a position, so it must be positive: $n = 11$

Answer: The 11th term

Tip: Reject $n = -6$: a term's position must be a positive whole number. Don't write $\pm$ in your answer.

Marks: 1 mark for $-2n^2 + 10n + 1 = -131$ rearranged to a quadratic $= 0$. 1 mark for 11 only.

5 (3 marks)

Here are the first five terms of a quadratic sequence.

3 7.5 13 19.5 27

Work out an expression for the n th term.

First differences: 4.5, 5.5, 6.5, 7.5

Second difference: 1, so the n^2 term is $\frac{1}{2}n^2$

Subtract $\frac{1}{2}n^2$ from each term:

$$3 - 0.5 = 2.5, \quad 7.5 - 2 = 5.5, \quad 13 - 4.5 = 8.5, \quad 19.5 - 8 = 11.5$$

2.5, 5.5, 8.5, 11.5 has n th term $3n - 0.5$

$$n\text{th term} = \frac{1}{2}n^2 + 3n - \frac{1}{2}$$

Check $n = 5$: $12.5 + 15 - 0.5 = 27$ ✓

Answer: $\frac{1}{2}n^2 + 3n - \frac{1}{2}$ (or $\frac{n^2 + 6n - 1}{2}$)

Tip: A second difference of 1 gives $\frac{1}{2}n^2$, not n^2 . Fractions or decimals in the n th term are fine.

Marks: 1 mark for $\frac{1}{2}n^2$. 1 mark for $3n$ or $-\frac{1}{2}$ (from a correct method). 1 mark for a fully correct expression.

6 (5 marks)

The n th term of a sequence is $\frac{4n - 3}{2n + 5}$

(a) Show that the 10th term is $\frac{26}{75}$ more than the 5th term.

$$\text{10th term: } \frac{4 \times 10 - 3}{2 \times 10 + 5} = \frac{37}{25}$$

$$\text{5th term: } \frac{4 \times 5 - 3}{2 \times 5 + 5} = \frac{17}{15}$$

$$\text{Difference: } \frac{37}{25} - \frac{17}{15} = \frac{111}{75} - \frac{85}{75} = \frac{26}{75}$$

Answer: $\frac{37}{25} - \frac{17}{15} = \frac{26}{75}$

Tip: In a “show that” question, write down each term and show the subtraction with a common denominator. Just writing $\frac{26}{75}$ is not enough.

Marks: 1 mark for both terms, $\frac{37}{25}$ and $\frac{17}{15}$. 1 mark for showing the subtraction with a common denominator to get $\frac{26}{75}$.

(b) Which term of the sequence has the value $\frac{5}{3}$?

$$\text{Solve } \frac{4n - 3}{2n + 5} = \frac{5}{3}$$

$$\text{Cross-multiply: } 3(4n - 3) = 5(2n + 5)$$

$$12n - 9 = 10n + 25$$

$$2n = 34$$

$$n = 17$$

Check: $\frac{68 - 3}{34 + 5} = \frac{65}{39} = \frac{5}{3} \checkmark$

Answer: The 17th term

Tip: Cross-multiply as one equation. Setting the tops equal ($4n - 3 = 5$) and the bottoms equal does not work here, because $\frac{65}{39}$ cancels to $\frac{5}{3}$.

Marks: 1 mark for $3(4n - 3) = 5(2n + 5)$. 1 mark for 17.

- (c) Write down the limiting value of the sequence as $n \rightarrow \infty$

For large n , the -3 and the $+5$ hardly matter.

So the terms get closer and closer to $\frac{4n}{2n} = 2$

Answer: 2

Tip: Compare the coefficients of n on the top and bottom: $\frac{4}{2} = 2$.

7 (5 marks)

The n th term of a quadratic sequence is $an^2 + bn + 7$, where a and b are constants.

The 2nd term of the sequence is 1

The 5th term of the sequence is -23

- (a) Work out the values of a and b .

2nd term: $4a + 2b + 7 = 1$, so $4a + 2b = -6$, which simplifies to $2a + b = -3$

5th term: $25a + 5b + 7 = -23$, so $25a + 5b = -30$, which simplifies to $5a + b = -6$

Subtract the first equation from the second: $3a = -3$, so $a = -1$

Substitute: $2(-1) + b = -3$, so $b = -1$

Check: n th term $= -n^2 - n + 7$. When $n = 5$: $-25 - 5 + 7 = -23$ ✓

Answer: $a = -1$, $b = -1$

Tip: Substitute $n = 2$ and $n = 5$ carefully: $a \times 5^2 = 25a$. Then check your values in both terms before going on.

Marks: 1 mark for one correct equation in a and b . 1 mark for a correct method to solve the two equations. 1 mark for both values.

- (b) Which term of the sequence has the value -83 ?

Do not use trial and improvement.

Solve $-n^2 - n + 7 = -83$

$$-n^2 - n + 90 = 0$$

Multiply by -1 : $n^2 + n - 90 = 0$

Factorise: $(n + 10)(n - 9) = 0$

$n = 9$ (reject $n = -10$, since a position must be positive)

Check: $-81 - 9 + 7 = -83$ ✓

Answer: The 9th term

Tip: Try to factorise before using the formula: $90 = 10 \times 9$. Give the position, 9, not ± 9 or both solutions.

Marks: 1 mark for a correct quadratic equation $= 0$ using your a and b . 1 mark for 9.

8 (5 marks)

The n th term of a sequence is $\frac{an + 3}{n + b}$, where a and b are constants.

The limiting value of the sequence as $n \rightarrow \infty$ is 4

- (a) Write down the value of a .

For large n , the terms get close to $\frac{an}{n} = a$

The limiting value is 4, so $a = 4$.

Answer: $a = 4$

- (b) The 3rd term of the sequence is 1.875

Work out the value of b .

$$\text{3rd term: } \frac{4 \times 3 + 3}{3 + b} = \frac{15}{3 + b}$$

$$\text{So } \frac{15}{3 + b} = 1.875$$

$$3 + b = \frac{15}{1.875} = 8$$

$$b = 5$$

Answer: $b = 5$

Marks: 1 mark for $\frac{15}{3+b} = 1.875$ (using your a). 1 mark for $b = 5$.

- (c) Work out the position of the first term of the sequence that is greater than 3.8
You must show your working.

$$\text{The } n\text{th term is } \frac{4n + 3}{n + 5}$$

$$\text{Solve } \frac{4n + 3}{n + 5} > 3.8$$

$n + 5$ is positive, so multiply both sides by $n + 5$: $4n + 3 > 3.8(n + 5)$

$$4n + 3 > 3.8n + 19$$

$$0.2n > 16$$

$$n > 80$$

The 80th term is exactly $\frac{323}{85} = 3.8$, so the first term greater than 3.8 is the 81st.

Answer: 81st term

Tip: $n > 80$ means the 80th term is not included. Check by substituting: the 80th term equals 3.8, and the 81st is just above it.

Marks: 1 mark for $4n + 3 > 3.8(n + 5)$ (or $=$). 1 mark for 81.