

Pythagoras and Trigonometry in 3D

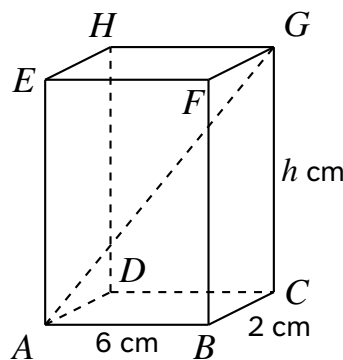
Solutions

For AQA Level 2 Certificate in Further Mathematics (8365)

Each question is shown in grey, followed by the working, one step per line. The final answer is in a box. *Marks* shows how the marks are usually given in the exam, and **Tip** points out a common mistake.

1 (2 marks)

ABCDEFGH is a cuboid.



Not drawn accurately

AB = 6 cm, BC = 2 cm and CG = h cm

The diagonal AG = 11 cm

Work out the value of h.

3D Pythagoras: $AG^2 = AB^2 + BC^2 + CG^2$

$$11^2 = 6^2 + 2^2 + h^2$$

$$121 = 40 + h^2, \text{ so } h^2 = 81$$

$$h = \sqrt{81} = 9$$

Answer: $h = 9$

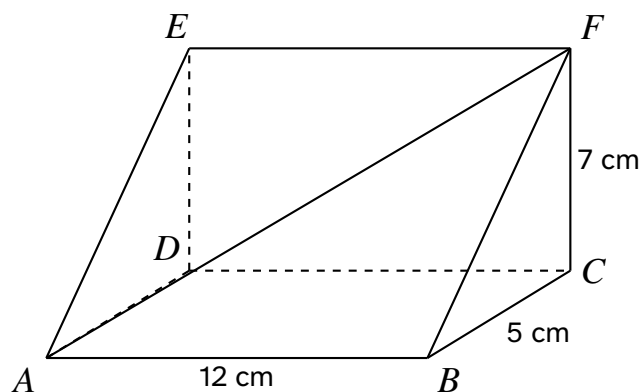
Tip: Square the 11 and subtract: $h^2 = 121 - 36 - 4$. Simplify the root at the end: $\sqrt{81} = 9$.

Marks: 1 mark for $6^2 + 2^2 + h^2 = 11^2$. 1 mark for 9.

2 (4 marks)

The diagram shows a wedge.

The base $ABCD$ is a horizontal rectangle and the face $CDEF$ is a vertical rectangle.



Not drawn accurately

- (a) Work out the length AF .

C is directly below F .

Base diagonal: $AC^2 = 12^2 + 5^2 = 169$, so $AC = 13$ cm

Triangle ACF has a right angle at C : $AF^2 = 13^2 + 7^2 = 218$

$AF = \sqrt{218} = 14.76 \dots$

Answer: 14.8 cm

Marks: 1 mark for $12^2 + 5^2 + 7^2$ (or $13^2 + 7^2$). 1 mark for 14.8.

- (b) Work out the angle between AF and the base $ABCD$.

C is directly below F , so the “shadow” of AF on the base is AC .

The angle we want is angle FAC .

In triangle ACF : opposite = $CF = 7$, adjacent = $AC = 13$

$$\tan FAC = \frac{7}{13}$$

$$\text{Angle } FAC = \tan^{-1}\left(\frac{7}{13}\right) = 28.30 \dots$$

Answer: 28.3°

Tip: Draw triangle ACF on its own. The angle is at A , between AF and the base diagonal AC , not between AF and AB .

Marks: 1 mark for $\tan \theta = \frac{7}{13}$ (or $\sin \theta = \frac{7}{\sqrt{218}}$). 1 mark for 28.3.

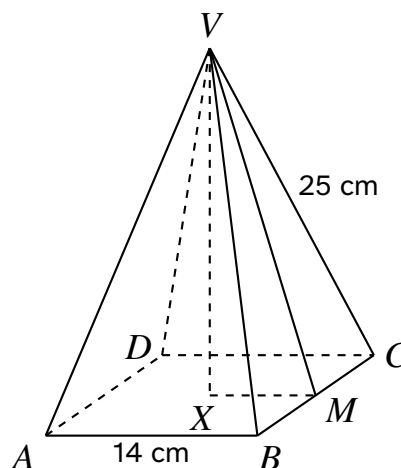
3 (4 marks)

$VABCD$ is a pyramid with a square base $ABCD$ of side 14 cm.

X is the centre of the base and V is vertically above X .

Each sloping edge of the pyramid is 25 cm long.

M is the midpoint of BC .



Not drawn accurately

- (a) Show that $VM = 24$ cm

Triangle VBC is isosceles ($VB = VC = 25$), so VM is perpendicular to BC .

$$BM = \frac{1}{2} \times 14 = 7 \text{ cm}$$

$$VM^2 = VB^2 - BM^2 = 25^2 - 7^2 = 625 - 49 = 576$$

$$VM = \sqrt{576} = 24 \text{ cm } \checkmark$$

Answer: Shown: $VM = \sqrt{625 - 49} = 24$

Tip: M is the midpoint of BC , so use $BM = 7$ cm, not 14 cm. (7, 24, 25 is a Pythagorean triple.)

Marks: 1 mark for $25^2 - 7^2$. 1 mark for $\sqrt{576} = 24$.

(b) Work out the angle between the face VBC and the base $ABCD$.

VM and XM are both perpendicular to BC , so the angle between the face and the base is angle VMX .

$$XM = \frac{1}{2} \times 14 = 7 \text{ cm}$$

Triangle VXM has a right angle at X , with hypotenuse $VM = 24$ and adjacent $XM = 7$

$$\cos VMX = \frac{7}{24}$$

$$\text{Angle } VMX = \cos^{-1}\left(\frac{7}{24}\right) = 73.04 \dots$$

Answer: 73.0°

Tip: The angle between two planes is measured with two lines that are both perpendicular to the edge where the planes meet (BC). Here they are VM and XM .

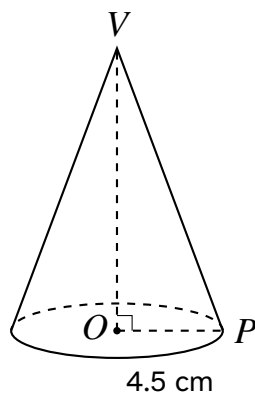
Marks: 1 mark for $\cos \theta = \frac{7}{24}$ (or another correct ratio). 1 mark for 73.0.

4 (4 marks)

The diagram shows a cone with vertex V .

O is the centre of the base and P is a point on the edge of the base.

The radius of the base is 4.5 cm.



Not drawn accurately

The volume of the cone is $81\pi \text{ cm}^3$

Work out the angle between the slant edge VP and the base of the cone.

[Volume of a cone = $\frac{1}{3}\pi r^2 h$, where r is the radius and h is the perpendicular height]

Step 1: find the height VO .

$$\frac{1}{3}\pi \times 4.5^2 \times h = 81\pi$$

$$6.75h = 81, \text{ so } h = 12 \text{ cm}$$

Step 2: triangle VOP has a right angle at O , and the angle we want is angle VPO .

Opposite = $VO = 12$, adjacent = $OP = 4.5$

$$\tan VPO = \frac{12}{4.5}$$

$$\text{Angle } VPO = \tan^{-1}\left(\frac{12}{4.5}\right) = 69.44 \dots$$

Answer: 69.4°

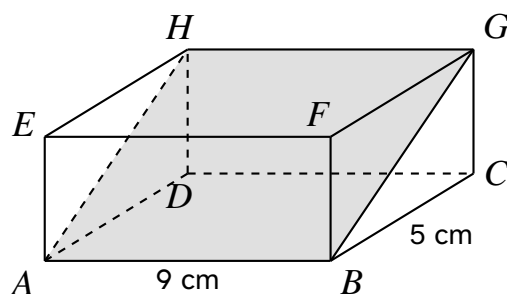
Tip: The π cancels on both sides, so you don't need a decimal for 81π . Then use the radius (4.5 cm), not the diameter.

Marks: 1 mark for $\frac{1}{3}\pi \times 4.5^2 \times h = 81\pi$. 1 mark for $h = 12$. 1 mark for $\tan \theta = \frac{12}{4.5}$ (or another correct ratio). 1 mark for 69.4.

5 (3 marks)

$ABCDEFGH$ is a cuboid.

The rectangle $ABGH$ is shaded.



Not drawn accurately

The angle between the plane $ABGH$ and the base $ABCD$ is 38°

Work out the area of the rectangle $ABGH$.

AB is the edge where the two planes meet.

BC (in the base) and BG (in the plane $ABGH$) are both perpendicular to AB , so the angle between the planes is angle $GBC = 38^\circ$.

Triangle BCG has a right angle at C , with adjacent $BC = 5$ and hypotenuse BG .

$$\cos 38^\circ = \frac{5}{BG}, \text{ so } BG = \frac{5}{\cos 38^\circ} = 6.345 \dots \text{ cm}$$

$$\text{Area of } ABGH = AB \times BG = 9 \times 6.345 \dots = 57.10 \dots$$

Answer: 57.1 cm²

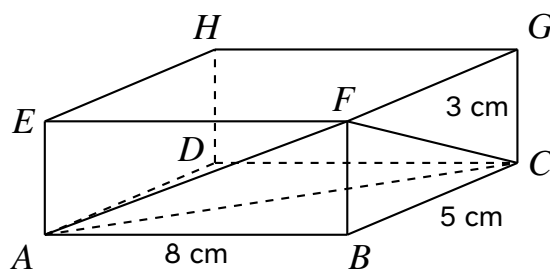
Tip: Sketch triangle BCG on its own and mark the 38° at B . Many students put the angle in the wrong place, for example at G instead of at B .

Marks: 1 mark for identifying angle $GBC = 38^\circ$. 1 mark for $BG = \frac{5}{\cos 38^\circ}$. 1 mark for 57.1.

6 (5 marks)

$ABCDEFGH$ is a cuboid.

Triangle AFC is drawn inside the cuboid.



Not drawn accurately

Work out the size of angle AFC .

Triangle AFC is not right-angled, so find all three sides, then use the cosine rule.

$$AF^2 = 8^2 + 3^2 = 73 \text{ (front face)}$$

$$FC^2 = 5^2 + 3^2 = 34 \text{ (right-hand face)}$$

$$AC^2 = 8^2 + 5^2 = 89 \text{ (base)}$$

Cosine rule with AC opposite angle F : $89 = 73 + 34 - 2\sqrt{73}\sqrt{34}\cos F$

$$\cos F = \frac{73 + 34 - 89}{2\sqrt{73}\sqrt{34}} = \frac{18}{2\sqrt{2482}} = 0.1806 \dots$$

$$F = \cos^{-1}(0.1806 \dots) = 79.59 \dots$$

Answer: 79.6°

Tip: The side opposite the angle you want (AC) goes on its own in $a^2 = b^2 + c^2 - 2bc \cos A$.
Keep the squared lengths exact (73, 34, 89) to avoid rounding errors.

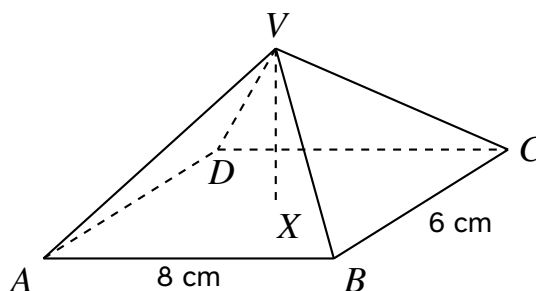
Marks: 1 mark for each of AF , FC and AC (3 marks). 1 mark for a correct cosine rule with AC opposite the angle. 1 mark for 79.6.

7 (5 marks)

$VABCD$ is a pyramid.

The base $ABCD$ is a horizontal rectangle.

X is the centre of the base and V is vertically above X .



Not drawn accurately

The angle between the face VAB and the base $ABCD$ is 55°

Work out the angle between the edge VA and the base $ABCD$.

Step 1: let N be the midpoint of AB . VN and XN are both perpendicular to AB , so angle $VNX = 55^\circ$.

$$XN = \frac{1}{2} \times BC = 3 \text{ cm}$$

Triangle VXN has a right angle at X : $VX = 3 \tan 55^\circ = 4.284 \dots \text{ cm}$

Step 2: X is the midpoint of the diagonal AC .

$$AC = \sqrt{8^2 + 6^2} = 10 \text{ cm, so } AX = 5 \text{ cm}$$

Step 3: the angle between VA and the base is angle VAX .

$$\tan VAX = \frac{VX}{AX} = \frac{4.284 \dots}{5}$$

$$\text{Angle } VAX = \tan^{-1}(0.8568 \dots) = 40.59 \dots$$

Answer: 40.6°

Tip: XN is half of BC (3 cm), not half of AB . And AX is half the diagonal, not half a side. Keep VX unrounded.

Marks: 1 mark for $XN = 3$. 1 mark for $VX = 3 \tan 55^\circ$. 1 mark for $AX = 5$. 1 mark for $\tan \theta = \frac{VX}{AX}$.
1 mark for 40.6.