

Pythagoras and Trigonometry in 2D

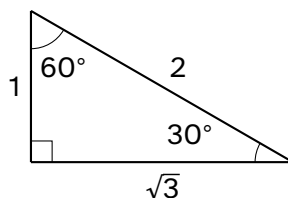
Solutions

For AQA Level 2 Certificate in Further Mathematics (8365)

Each question is shown in grey, followed by the working, one step per line. The final answer is in a box. *Marks* shows how the marks are usually given in the exam, and **Tip** points out a common mistake.

1 (3 marks)

The diagram shows a right-angled triangle with sides 1, $\sqrt{3}$ and 2



Not drawn accurately

(a) Circle the value of $\tan 30^\circ \div \sin 60^\circ$

Options: $\frac{1}{2}$, $\frac{2}{3}$, $\frac{\sqrt{3}}{3}$, $\frac{3}{2}$

From the triangle: $\tan 30^\circ = \frac{\text{opp}}{\text{adj}} = \frac{1}{\sqrt{3}}$ and $\sin 60^\circ = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{3}}{2}$

$$\frac{1}{\sqrt{3}} \div \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{3}} \times \frac{2}{\sqrt{3}} = \frac{2}{3}$$

Answer: $\frac{2}{3}$

Tip: Learn the exact values for 30° , 45° and 60° , or sketch this triangle and the 1:1: $\sqrt{2}$ triangle to work them out. Remember $\sqrt{3} \times \sqrt{3} = 3$.

(b) Work out the exact value of $6 \sin 60^\circ - 2 \tan 30^\circ$

Give your answer in the form $\frac{a\sqrt{3}}{b}$, where a and b are integers.

$$6 \sin 60^\circ = 6 \times \frac{\sqrt{3}}{2} = 3\sqrt{3}$$

$$2 \tan 30^\circ = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \text{ (rationalise)}$$

$$3\sqrt{3} - \frac{2\sqrt{3}}{3} = \frac{9\sqrt{3}}{3} - \frac{2\sqrt{3}}{3} = \frac{7\sqrt{3}}{3}$$

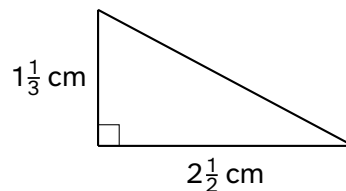
Answer: $\frac{7\sqrt{3}}{3}$

Tip: Rationalise $\frac{2}{\sqrt{3}}$ first, so both terms are thirds of $\sqrt{3}$ and can be combined.

Marks: 1 mark for $3\sqrt{3}$ and $\frac{2}{\sqrt{3}}$ (or $\frac{2\sqrt{3}}{3}$). 1 mark for $\frac{7\sqrt{3}}{3}$.

2 (2 marks)

Here is a right-angled triangle.



Not drawn accurately

Work out the length of the hypotenuse.

Give your answer as a mixed number.

Write the lengths as improper fractions: $1\frac{1}{3} = \frac{4}{3}$ and $2\frac{1}{2} = \frac{5}{2}$

$$h^2 = \left(\frac{4}{3}\right)^2 + \left(\frac{5}{2}\right)^2 = \frac{16}{9} + \frac{25}{4}$$

$$= \frac{64}{36} + \frac{225}{36} = \frac{289}{36}$$

$$h = \frac{\sqrt{289}}{\sqrt{36}} = \frac{17}{6} = 2\frac{5}{6}$$

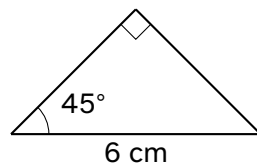
Answer: $2\frac{5}{6}$ cm

Tip: Stay in fractions. Look for a Pythagorean triple: $\frac{8}{6}$, $\frac{15}{6}$ and $\frac{17}{6}$ are 8, 15, 17 divided by 6.

Marks: 1 mark for $\left(\frac{4}{3}\right)^2 + \left(\frac{5}{2}\right)^2$. 1 mark for $2\frac{5}{6}$.

3 (2 marks)

Here is a right-angled triangle.



Not drawn accurately

Work out the area of the triangle.

The triangle has angles 45°, 45° and 90°, so the two shorter sides are equal.

Sides are in the ratio 1:1: $\sqrt{2}$, so each shorter side is $\frac{6}{\sqrt{2}} = 3\sqrt{2}$ cm

$$\text{Area} = \frac{1}{2} \times 3\sqrt{2} \times 3\sqrt{2} = \frac{1}{2} \times 18 = 9$$

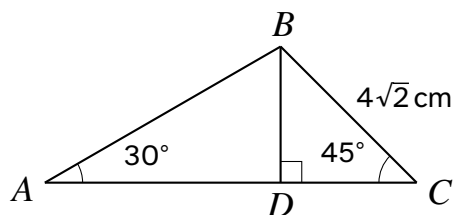
Answer: 9 cm²

Tip: $3\sqrt{2} \times 3\sqrt{2} = 9 \times 2 = 18$. Multiply the whole numbers and the roots separately.

Marks: 1 mark for a shorter side of $\frac{6}{\sqrt{2}}$ or $3\sqrt{2}$ (or $\sqrt{18}$). 1 mark for 9.

4 (4 marks)

In triangle ABC , BD is perpendicular to AC .



Not drawn accurately

Work out the exact perimeter of triangle ABC .

Give your answer in the form $a + b\sqrt{2} + c\sqrt{3}$, where a , b and c are integers.

Triangle BDC : $BD = 4\sqrt{2} \times \sin 45^\circ = 4\sqrt{2} \times \frac{\sqrt{2}}{2} = 4$ cm, and $DC = BD = 4$ cm (45°, 45°, 90° triangle)

Triangle ABD: $\sin 30^\circ = \frac{BD}{AB}$, so $AB = \frac{4}{\frac{1}{2}} = 8 \text{ cm}$

$\tan 30^\circ = \frac{BD}{AD}$, so $AD = \frac{4}{\tan 30^\circ} = 4 \div \frac{1}{\sqrt{3}} = 4\sqrt{3} \text{ cm}$

Perimeter = $AB + BC + AD + DC = 8 + 4\sqrt{2} + 4\sqrt{3} + 4$
 $= 12 + 4\sqrt{2} + 4\sqrt{3}$

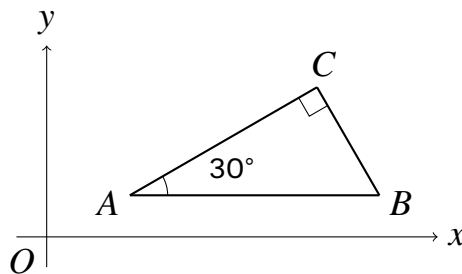
Answer: $12 + 4\sqrt{2} + 4\sqrt{3} \text{ cm}$

Tip: Work in the right-angled triangles BDC and ABD . The perimeter uses $AC = AD + DC$, not BD .

Marks: 1 mark for $BD = 4$ (or $DC = 4$). 1 mark for $AB = 8$. 1 mark for $AD = 4\sqrt{3}$. 1 mark for the correct perimeter.

5 (3 marks)

Triangle ABC has vertices $A(2, 1)$, $B(8, 1)$ and C .
 Angle $ACB = 90^\circ$ and angle $CAB = 30^\circ$



Not drawn accurately

Work out the exact coordinates of C .

$AB = 8 - 2 = 6$

In triangle ABC (right angle at C): $AC = AB \cos 30^\circ = 6 \times \frac{\sqrt{3}}{2} = 3\sqrt{3}$

Drop a perpendicular from C to AB , meeting it at F .

Across: $AF = AC \cos 30^\circ = 3\sqrt{3} \times \frac{\sqrt{3}}{2} = \frac{9}{2}$

Up: $CF = AC \sin 30^\circ = 3\sqrt{3} \times \frac{1}{2} = \frac{3\sqrt{3}}{2}$

$$C = \left(2 + \frac{9}{2}, 1 + \frac{3\sqrt{3}}{2} \right) = \left(\frac{13}{2}, 1 + \frac{3\sqrt{3}}{2} \right)$$

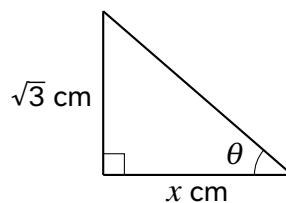
Answer: $\left(\frac{13}{2}, 1 + \frac{3\sqrt{3}}{2} \right)$

Tip: Add on the coordinates of A at the end: $\frac{9}{2}$ and $\frac{3\sqrt{3}}{2}$ are distances from A , not the coordinates of C .

Marks: 1 mark for $AC = 3\sqrt{3}$. 1 mark for $\frac{9}{2}$ across or $\frac{3\sqrt{3}}{2}$ up. 1 mark for both coordinates.

6 (3 marks)

Here is a right-angled triangle.



Not drawn accurately

$$1 < x < 3$$

Work out the range of possible values of θ .

You must show your working.

$$\tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{3}}{x}$$

When $x = 1$: $\tan \theta = \sqrt{3}$, so $\theta = 60^\circ$

When $x = 3$: $\tan \theta = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$, so $\theta = 30^\circ$

As x gets bigger, $\frac{\sqrt{3}}{x}$ gets smaller, so θ gets smaller.

x is between 1 and 3 (not equal to either), so θ is between 30° and 60° .

Answer: $30^\circ < \theta < 60^\circ$

Tip: Don't just try one value of x . Find θ at both ends and think about which way θ moves: a longer adjacent side makes the angle smaller.

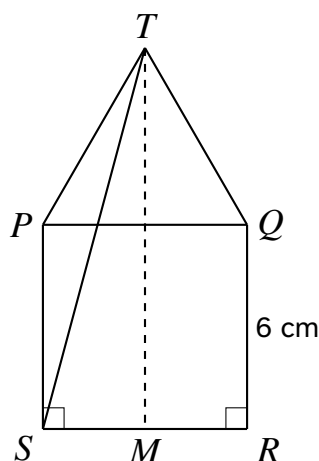
Marks: 1 mark for $\tan \theta = \frac{\sqrt{3}}{x}$. 1 mark for 30° and 60° . 1 mark for $30^\circ < \theta < 60^\circ$.

7 (5 marks)

The diagram shows a pentagon made from a square $PQRS$ and an equilateral triangle PQT .

The square has side length 6 cm.

M is the midpoint of SR .



Not drawn accurately

- (a) Show that $TM = (6 + 3\sqrt{3})$ cm

Let N be the midpoint of PQ . TN is the line of symmetry of the equilateral triangle, so TN is perpendicular to PQ and $PN = 3$ cm.

Every angle of an equilateral triangle is 60° , so in right-angled triangle PNT :

$$TN = PT \sin 60^\circ = 6 \times \frac{\sqrt{3}}{2} = 3\sqrt{3} \text{ cm}$$

(Or Pythagoras: $TN = \sqrt{6^2 - 3^2} = \sqrt{27} = 3\sqrt{3}$ cm)

T , N and M all lie on the line of symmetry of the pentagon, and $NM = 6$ cm (a side of the square).

$$TM = TN + NM = 3\sqrt{3} + 6 = (6 + 3\sqrt{3}) \text{ cm } \checkmark$$

Answer: Shown: $TM = 3\sqrt{3} + 6$

Tip: Split the equilateral triangle down its line of symmetry: you get a 30° - 60° - 90° triangle with hypotenuse 6 cm and shortest side 3 cm. In a 'show that', write $\sqrt{27} = 3\sqrt{3}$ as a step.

Marks: 1 mark for the height of the triangle, $3\sqrt{3}$ (or $6 \sin 60^\circ$ or $\sqrt{27}$). 1 mark for adding the 6 cm side of the square to reach the given answer.

(b) Work out ST^2

Give your answer in the form $a + b\sqrt{3}$, where a and b are integers.

Triangle SMT has a right angle at M , with $SM = 3$ cm (half of SR) and $TM = (6 + 3\sqrt{3})$ cm.

$$\text{Pythagoras: } ST^2 = SM^2 + TM^2 = 3^2 + (6 + 3\sqrt{3})^2$$

$$(6 + 3\sqrt{3})^2 = (6 + 3\sqrt{3})(6 + 3\sqrt{3}) = 36 + 18\sqrt{3} + 18\sqrt{3} + 27$$

(The last term: $3\sqrt{3} \times 3\sqrt{3} = 9 \times 3 = 27$)

$$= 63 + 36\sqrt{3}$$

$$ST^2 = 9 + 63 + 36\sqrt{3} = 72 + 36\sqrt{3}$$

Answer: $ST^2 = 72 + 36\sqrt{3}$

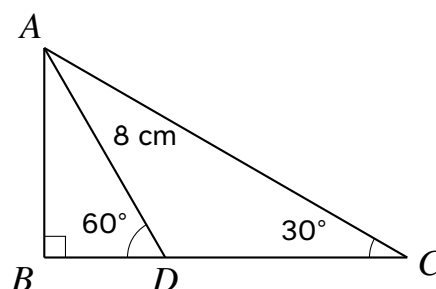
Tip: $(6 + 3\sqrt{3})^2$ is not $36 + 27$: write out both brackets and multiply every term, so there are two middle terms of $18\sqrt{3}$. Also $(3\sqrt{3})^2 = 9 \times 3 = 27$, not $9\sqrt{3}$.

Marks: 1 mark for $3^2 + (6 + 3\sqrt{3})^2$. 1 mark for $(6 + 3\sqrt{3})^2 = 63 + 36\sqrt{3}$ (or four correct terms). 1 mark for $72 + 36\sqrt{3}$.

8 (4 marks)

ABC is a right-angled triangle.

D is a point on BC .



Not drawn accurately

Work out the exact area of triangle ADC .

Give your answer in the form $a\sqrt{3}$, where a is an integer.

Triangle ABD : $AB = 8 \sin 60^\circ = 8 \times \frac{\sqrt{3}}{2} = 4\sqrt{3} \text{ cm}$

$$BD = 8 \cos 60^\circ = 8 \times \frac{1}{2} = 4 \text{ cm}$$

Triangle ABC : $\tan 30^\circ = \frac{AB}{BC}$, so $BC = \frac{4\sqrt{3}}{\tan 30^\circ} = 4\sqrt{3} \div \frac{1}{\sqrt{3}} = 4\sqrt{3} \times \sqrt{3} = 12 \text{ cm}$

$$DC = BC - BD = 12 - 4 = 8 \text{ cm}$$

Triangle ADC has base $DC = 8$ and perpendicular height $AB = 4\sqrt{3}$.

$$\text{Area} = \frac{1}{2} \times 8 \times 4\sqrt{3} = 16\sqrt{3}$$

Answer: $16\sqrt{3} \text{ cm}^2$

Tip: The height of triangle ADC is AB , outside the triangle. Also, angle $DAC = 60^\circ - 30^\circ = 30^\circ$, so triangle ADC is isosceles and $DC = AD = 8$: a quick check.

Marks: 1 mark for $AB = 4\sqrt{3}$. 1 mark for $BC = 12$ (or $BD = 4$ and $DC = 8$). 1 mark for $\frac{1}{2} \times DC \times AB$. 1 mark for $16\sqrt{3}$.