

Maximum and Minimum Problems

Solutions

For AQA Level 2 Certificate in Further Mathematics (8365)

Each question is shown in grey, followed by the working, one step per line. The final answer is in a box. *Marks* shows how the marks are usually given in the exam, and **Tip** points out a common mistake.

1 (3 marks)

A ball is thrown upwards from a balcony.

After t seconds, the height of the ball above the ground is h metres, where

$$h = 2 + 12t - 5t^2$$

Use calculus to work out the maximum height of the ball.

$$\frac{dh}{dt} = 12 - 10t$$

At the maximum, $\frac{dh}{dt} = 0$: $12 - 10t = 0$, so $t = 1.2$

Substitute $t = 1.2$ into the formula for h :

$$h = 2 + 12 \times 1.2 - 5 \times 1.2^2$$

$$= 2 + 14.4 - 7.2 = 9.2$$

Answer: 9.2 m

Tip: The question asks for the maximum **height**, not the time. Once you have $t = 1.2$, substitute it back into the formula for h .

Marks: 1 mark for $12 - 10t$. 1 mark for $t = 1.2$. 1 mark for 9.2.

2 (5 marks)

$$y = 5x + \frac{80}{x}, \text{ where } x > 0$$

(a) Work out the value of x for which $\frac{dy}{dx} = 0$

Write the fraction as a power of x : $y = 5x + 80x^{-1}$

$$\frac{dy}{dx} = 5 - 80x^{-2} = 5 - \frac{80}{x^2}$$

Set equal to zero: $5 - \frac{80}{x^2} = 0$, so $5x^2 = 80$

$x^2 = 16$, so $x = 4$ ($x = -4$ is not allowed because $x > 0$)

Answer: $x = 4$

Tip: $\frac{80}{x} = 80x^{-1}$ differentiates to $-80x^{-2}$: multiply by the power, then take 1 off the power.

Marks: 1 mark for $5 - 80x^{-2}$. 1 mark for $x = 4$.

- (b) Use $\frac{d^2y}{dx^2}$ to show that y has a minimum value at this value of x .
Work out the minimum value of y .

$$\frac{dy}{dx} = 5 - 80x^{-2}$$

$$\frac{d^2y}{dx^2} = 160x^{-3} = \frac{160}{x^3}$$

When $x = 4$: $\frac{d^2y}{dx^2} = \frac{160}{64} = 2.5$

$\frac{d^2y}{dx^2} > 0$, so y has a minimum value when $x = 4$.

Minimum value: $y = 5 \times 4 + \frac{80}{4} = 20 + 20 = 40$

Answer: $\frac{d^2y}{dx^2} = 2.5 > 0$, so it is a minimum. The minimum value of y is 40

Tip: $-80x^{-2}$ differentiates to $+160x^{-3}$ (the two minus signs make a plus). Positive second derivative means minimum.

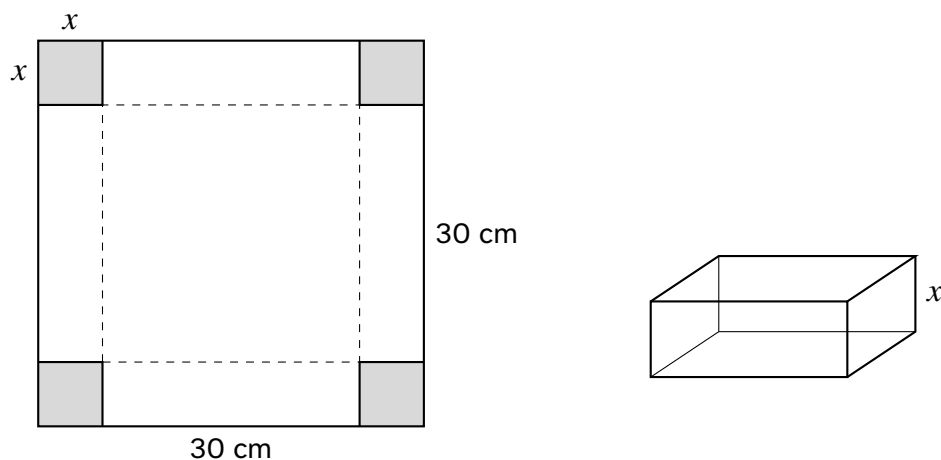
Marks: 1 mark for $160x^{-3}$. 1 mark for a positive value at $x = 4$ and the conclusion. 1 mark for 40.

3 (6 marks)

A square piece of card has sides of length 30 cm.

A square of side x cm is cut from each corner.

The card is then folded along the dashed lines to make an open box with no lid.



Not drawn accurately

- (a) Show that the volume of the box, $V \text{ cm}^3$, is given by

$$V = 4x^3 - 120x^2 + 900x$$

When the sides are folded up, the height of the box is x .

Each side of the square base is $30 - x - x = 30 - 2x$

$$V = x(30 - 2x)^2$$

$$(30 - 2x)^2 = 900 - 120x + 4x^2$$

$$V = x(900 - 120x + 4x^2) = 4x^3 - 120x^2 + 900x$$

Answer: $V = x(30 - 2x)^2 = 4x^3 - 120x^2 + 900x$

Tip: $(30 - 2x)^2$ has a middle term: it is $900 - 120x + 4x^2$, not $900 + 4x^2$. Show the expansion clearly in a 'show that'.

Marks: 1 mark for $x(30 - 2x)^2$ or $x(30 - 2x)(30 - 2x)$. 1 mark for expanding correctly to the given answer.

- (b) Use calculus to work out the maximum volume of the box.

$$\frac{dV}{dx} = 12x^2 - 240x + 900$$

At a maximum, $\frac{dV}{dx} = 0$: $12x^2 - 240x + 900 = 0$

Divide by 12: $x^2 - 20x + 75 = 0$

$(x - 5)(x - 15) = 0$, so $x = 5$ or $x = 15$

$x = 15$ is not possible: the base would have side $30 - 30 = 0$ (the volume would be 0). So $x = 5$.

(Check: $\frac{d^2V}{dx^2} = 24x - 240 = -120 < 0$ when $x = 5$, so this is a maximum.)

$$V = 5 \times (30 - 10)^2 = 5 \times 400 = 2000$$

Answer: 2000 cm³

Tip: Give the maximum **volume**, not the value of x : substitute $x = 5$ back into V . Trying values of x is not 'using calculus' and gets no marks.

Marks: 1 mark for $12x^2 - 240x + 900$. 1 mark for setting it equal to 0. 1 mark for $x = 5$ (rejecting 15). 1 mark for 2000.

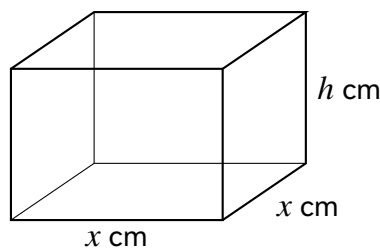
4 (6 marks)

A metal water tank is an open cuboid with no top.

The base is a square of side x cm. The height is h cm.

The volume of the tank is 4000 cm³.

The total area of metal used for the tank is A cm².



Not drawn accurately

(a) Show that $A = x^2 + \frac{16000}{x}$

Volume: $x \times x \times h = 4000$, so $h = \frac{4000}{x^2}$

The tank has a base and four sides (no top).

$$\text{Area of the base} = x^2$$

$$\text{Area of the four sides} = 4 \times xh = 4x \times \frac{4000}{x^2} = \frac{16000}{x}$$

$$A = x^2 + \frac{16000}{x}$$

Answer: $h = \frac{4000}{x^2}$, so $A = x^2 + 4xh = x^2 + \frac{16000}{x}$

Tip: There is no lid, so count only **one** square face (x^2), not two. Make sure no h is left in your final expression.

Marks: 1 mark for $h = \frac{4000}{x^2}$ (or $x^2h = 4000$). 1 mark for $x^2 + 4xh$ with h replaced, giving the answer.

- (b) Use calculus to work out the minimum value of A .
You must show that your value is a minimum.

$$A = x^2 + 16000x^{-1}$$

$$\frac{dA}{dx} = 2x - 16000x^{-2} = 2x - \frac{16000}{x^2}$$

$$\text{At a minimum, } \frac{dA}{dx} = 0: 2x = \frac{16000}{x^2}, \text{ so } x^3 = 8000$$

$$x = 20$$

$$\frac{d^2A}{dx^2} = 2 + 32000x^{-3} = 2 + \frac{32000}{x^3}$$

$$\text{When } x = 20: \frac{d^2A}{dx^2} = 2 + \frac{32000}{8000} = 6 > 0, \text{ so this is a minimum.}$$

$$A = 20^2 + \frac{16000}{20} = 400 + 800 = 1200$$

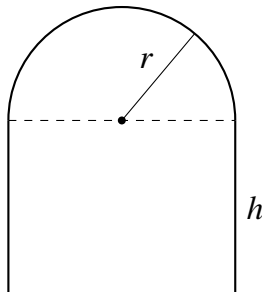
Answer: 1200 cm²

Tip: $x^3 = 8000$ needs a **cube** root, not a square root. Then substitute $x = 20$ into A : the answer is the area, not x .

Marks: 1 mark for $2x - 16000x^{-2}$. 1 mark for $x = 20$. 1 mark for a correct check that it is a minimum. 1 mark for 1200.

5 (6 marks)

A window is in the shape of a rectangle with a semicircle on top.
 The semicircle has radius r metres. The rectangle has height h metres.
 The frame around the outside edge of the window has a total length of 10 m.
 The area of the window is $A \text{ m}^2$.



Not drawn accurately

- (a) Show that $A = 10r - 2r^2 - \frac{1}{2}\pi r^2$

The frame is made of two sides of length h , the bottom of length $2r$, and the semicircle arc.

$$\text{Length of the semicircle arc} = \frac{1}{2} \times 2\pi r = \pi r$$

$$\text{So } 2h + 2r + \pi r = 10, \text{ which gives } 2h = 10 - 2r - \pi r$$

$$\text{Area of the rectangle} = 2r \times h = r \times 2h = r(10 - 2r - \pi r) = 10r - 2r^2 - \pi r^2$$

$$\text{Area of the semicircle} = \frac{1}{2}\pi r^2$$

$$A = 10r - 2r^2 - \pi r^2 + \frac{1}{2}\pi r^2 = 10r - 2r^2 - \frac{1}{2}\pi r^2$$

Answer: $2h = 10 - 2r - \pi r$, so $A = r(10 - 2r - \pi r) + \frac{1}{2}\pi r^2 = 10r - 2r^2 - \frac{1}{2}\pi r^2$

Tip: The frame goes round the **outside** only, so it does not include the dashed line across the middle. The arc is half the circumference: πr , not $2\pi r$.

Marks: 1 mark for $2h + 2r + \pi r = 10$. 1 mark for the rectangle area $2rh$ with h (or $2h$) replaced. 1 mark for adding $\frac{1}{2}\pi r^2$ and simplifying to the given answer.

- (b) Use calculus to work out the maximum area of the window.
 Give your answer to 3 significant figures.

$$\frac{dA}{dr} = 10 - 4r - \pi r$$

At a maximum, $\frac{dA}{dr} = 0$: $10 = 4r + \pi r = r(4 + \pi)$

$$r = \frac{10}{4 + \pi} = 1.4002 \dots$$

($\frac{d^2A}{dr^2} = -4 - \pi < 0$, so this is a maximum.)

$$A = 10(1.4002 \dots) - 2(1.4002 \dots)^2 - \frac{1}{2}\pi(1.4002 \dots)^2 = 7.0012 \dots$$

To 3 significant figures: 7.00

Answer: 7.00 m²

Tip: Keep the full calculator value of r (use the answer button) when you substitute. Rounding r to 1.4 first still gives 7.00 here, but it often costs the accuracy mark.

Marks: 1 mark for $10 = 4r + \pi r$. 1 mark for $r = 1.40 \dots$ (or $\frac{10}{4+\pi}$). 1 mark for 7.00.
