

Matrix Transformations

Solutions

For AQA Level 2 Certificate in Further Mathematics (8365)

Each question is shown in grey, followed by the working, one step per line. The final answer is in a box. *Marks* shows how the marks are usually given in the exam, and **Tip** points out a common mistake.

1 (1 mark)

Which matrix represents a rotation of 180° about the origin?

Circle your answer.

Options: $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Find where the points (1, 0) and (0, 1) go. Their images are the columns of the matrix.

A half turn about the origin takes (1, 0) to $(-1, 0)$: first column $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$

It takes (0, 1) to $(0, -1)$: second column $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$

The matrix is $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

Answer: $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

Tip: You do not need to remember the matrices. Work out the images of (1, 0) and (0, 1) and write them as the two columns.

2 (2 marks)

Write down the matrix that represents

(a) a reflection in the y-axis

A reflection in the y-axis takes (1, 0) to $(-1, 0)$ and leaves (0, 1) where it is.

Columns: $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Answer: $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Tip: The y-axis is the line $x = 0$. A common slip is to give the matrix for a reflection in the x-axis, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, instead.

(b) an enlargement, scale factor 4, centre the origin

The enlargement takes $(1, 0)$ to $(4, 0)$ and $(0, 1)$ to $(0, 4)$.

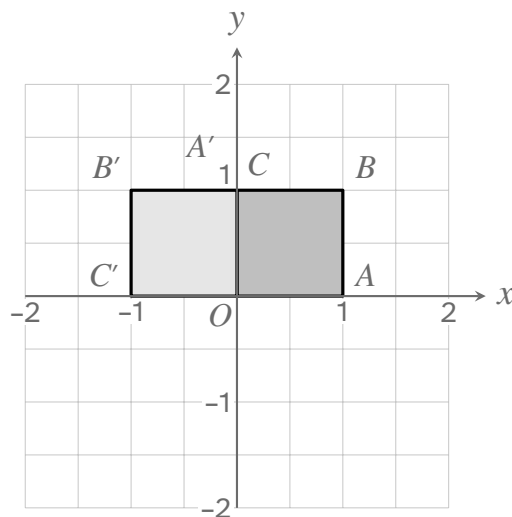
Columns: $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$

Answer: $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

3 (3 marks)

$OABC$ is the unit square, with $A(1,0)$, $B(1,1)$ and $C(0,1)$

$$\mathbf{T} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



- (a) On the grid, draw the image of $OABC$ under the transformation represented by $\mathbf{T}$.

Multiply each vertex by $\mathbf{T}$ (matrix on the left, point as a column on the right).

$$A: \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ so } A' \text{ is } (0,1)$$

$$B: \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \text{ so } B' \text{ is } (-1,1)$$

$$C: \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \text{ so } C' \text{ is } (-1,0)$$

O stays at $(0,0)$. The image is shown on the grid above.

Answer: Square with vertices $(0,0)$, $(0,1)$, $(-1,1)$, $(-1,0)$

Tip: Shortcut: the columns of $\mathbf{T}$ are the images of $A(1,0)$ and $C(0,1)$. Then B' completes the square.

Marks: 1 mark for two correct image vertices. 1 mark for the correct square.

- (b) Describe fully the single transformation represented by **T**.

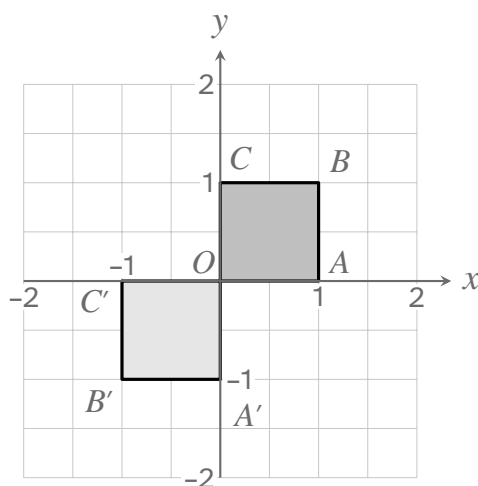
The square has turned a quarter turn anticlockwise about *O*: *A* on the positive *x*-axis has moved to the positive *y*-axis.

Answer: Rotation, 90° anticlockwise, about the origin

Tip: A rotation needs the angle, the direction and the centre. Write 'about the origin' or 'centre (0, 0)', not 'about the centre'.

4 (2 marks)

The grid shows the unit square *OABC* and its image *OA'B'C'* under the transformation represented by matrix **M**.



- (a) Write down matrix **M**

The first column is the image of *A*(1,0): *A'* is (0,−1)

The second column is the image of *C*(0,1): *C'* is (−1,0)

$$\mathbf{M} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

Check B : $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$, which is B' ✓

Answer: $\mathbf{M} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

Tip: Read the images of A and C carefully: A' gives the **first** column and C' the **second**. Putting the -1 s and 0 s in the wrong places is the most common mistake.

- (b) Describe fully the single transformation represented by $\mathbf{M}$.

The mirror line passes through O , half-way between $A(1,0)$ and $A'(0,-1)$, and half-way between $C(0,1)$ and $C'(-1,0)$.

That is the line $y = -x$.

Check: $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ -x \end{pmatrix}$, so (x,y) goes to $(-y,-x)$, which is a reflection in $y = -x$

Answer: Reflection in the line $y = -x$

Tip: Give the equation of the mirror line, $y = -x$. 'Reflection in the diagonal' is not enough.

5 (4 marks)

Triangle DEF has vertices $D(1,2)$, $E(3,2)$ and $F(3,-1)$

DEF is transformed by the matrix $\mathbf{N} = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$ to triangle $D'E'F'$.

- (a) Work out the coordinates of D' , E' and F' .

Multiply each point by $\mathbf{N}$, with the point written as a column.

$$D: \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} \text{ so } D'(6,3)$$

$$E: \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \end{pmatrix} \text{ so } E'(6,9)$$

$$F: \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 9 \end{pmatrix} \text{ so } F'(-3, 9)$$

Answer: $D'(6, 3)$, $E'(6, 9)$, $F'(-3, 9)$

Tip: The matrix goes on the left: $\mathbf{N} \begin{pmatrix} x \\ y \end{pmatrix}$, not the other way round (a 2×1 matrix times a 2×2 matrix cannot be worked out).

Marks: 1 mark for one correct image. 1 mark for all three correct.

(b) Work out the area of triangle $D'E'F'$.

$D'(6, 3)$ and $E'(6, 9)$ have the same x -coordinate, so $D'E'$ is vertical with length $9 - 3 = 6$

$E'(6, 9)$ and $F'(-3, 9)$ have the same y -coordinate, so $E'F'$ is horizontal with length $6 - (-3) = 9$

So the triangle has a right angle at E' .

$$\text{Area} = \frac{1}{2} \times 6 \times 9 = 27$$

(Or: $\begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} = 3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, a reflection in $y = x$ with an enlargement, scale factor 3.

Area of $DEF = \frac{1}{2} \times 2 \times 3 = 3$, and areas are multiplied by $3^2 = 9$: $3 \times 9 = 27$)

Answer: 27 square units

Tip: If you use the scale factor, remember that areas are multiplied by the scale factor **squared**: 9, not 3.

Marks: 1 mark for the two perpendicular sides 6 and 9 (or for the area of $DEF = 3$ and the scale factor 3). 1 mark for 27.

6 (3 marks)

Matrix **D** represents a reflection in the x -axis.

Matrix **E** represents a reflection in the line $y = x$

(a) Work out **DE**

Reflection in the x -axis: $(1, 0)$ stays, $(0, 1) \rightarrow (0, -1)$, so $\mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Reflection in $y = x$: $(1, 0) \rightarrow (0, 1)$ and $(0, 1) \rightarrow (1, 0)$, so $\mathbf{E} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\mathbf{DE} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ (this means } \mathbf{E} \text{ first, then } \mathbf{D}\text{)}$$

Top row: $1 \times 0 + 0 \times 1 = 0$ and $1 \times 1 + 0 \times 0 = 1$

Bottom row: $0 \times 0 + (-1) \times 1 = -1$ and $0 \times 1 + (-1) \times 0 = 0$

$$\mathbf{DE} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Answer: $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Tip: Keep the order: $\mathbf{DE}$ means $\mathbf{D}$ on the left. $\mathbf{ED} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is a different matrix.

Marks: 1 mark for both matrices $\mathbf{D}$ and $\mathbf{E}$ correct. 1 mark for $\mathbf{DE}$.

- (b) Describe fully the single transformation represented by $\mathbf{DE}$.

$\mathbf{DE}$ takes $(1, 0)$ to $(0, -1)$ and $(0, 1)$ to $(1, 0)$

$(1, 0)$ on the positive x -axis moves to the negative y -axis: a quarter turn clockwise about O .

Answer: Rotation, 90° clockwise, about the origin

Tip: Always give the centre: 'about the origin'. Many students lose this mark by leaving it out.

7 (3 marks)

A reflection in the y -axis is followed by a rotation of 90° anticlockwise about the origin. Use matrix multiplication to show that the combined transformation is a reflection in the line $y = -x$

Reflection in the y -axis: $(1, 0) \rightarrow (-1, 0)$ and $(0, 1)$ stays, so its matrix is $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Rotation 90° anticlockwise: $(1, 0) \rightarrow (0, 1)$ and $(0, 1) \rightarrow (-1, 0)$, so its matrix is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The reflection happens first, so its matrix goes on the **right**:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 \times (-1) + (-1) \times 0 & 0 \times 0 + (-1) \times 1 \\ 1 \times (-1) + 0 \times 0 & 1 \times 0 + 0 \times 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ takes $(1, 0)$ to $(0, -1)$ and $(0, 1)$ to $(-1, 0)$, which is the matrix for a reflection in $y = -x$ ✓

Answer: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, the matrix for a reflection in $y = -x$

Tip: You must multiply the matrices: a drawing, or following one point, is not enough on its own. The transformation that happens second goes on the **left**.

Marks: 1 mark for both matrices correct. 1 mark for multiplying them in the correct order. 1 mark for $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ and stating it is a reflection in $y = -x$.

8 (4 marks)

Matrix **A** represents a reflection in the line $y = -x$

B is a 2×2 matrix.

The matrix **BA** represents a rotation of 180° about the origin.

(a) Work out matrix **B**

Reflection in $y = -x$: $(1, 0) \rightarrow (0, -1)$ and $(0, 1) \rightarrow (-1, 0)$, so **A** = $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

Rotation 180° about the origin: $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

Let **B** = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\mathbf{BA} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -b & -a \\ -d & -c \end{pmatrix}$$

$$\text{So } \begin{pmatrix} -b & -a \\ -d & -c \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$-b = -1, -a = 0, -d = 0, -c = -1$$

$$a = 0, b = 1, c = 1, d = 0$$

$$\mathbf{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\text{Check: } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \checkmark$$

$$\text{Answer: } \mathbf{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Tip: **BA** means **A** is applied first, then **B**. Keep **B** on the left when you multiply.

Marks: 1 mark for the matrices for **A** and for the rotation. 1 mark for a correct equation for

B (for example $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$). 1 mark for the correct matrix.

(b) Describe fully the single transformation represented by **B**.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ swaps } (1, 0) \text{ and } (0, 1).$$

(x, y) goes to (y, x) : a reflection in the line $y = x$

(This makes sense: reflecting in $y = -x$ and then in $y = x$ is a half turn about O .)

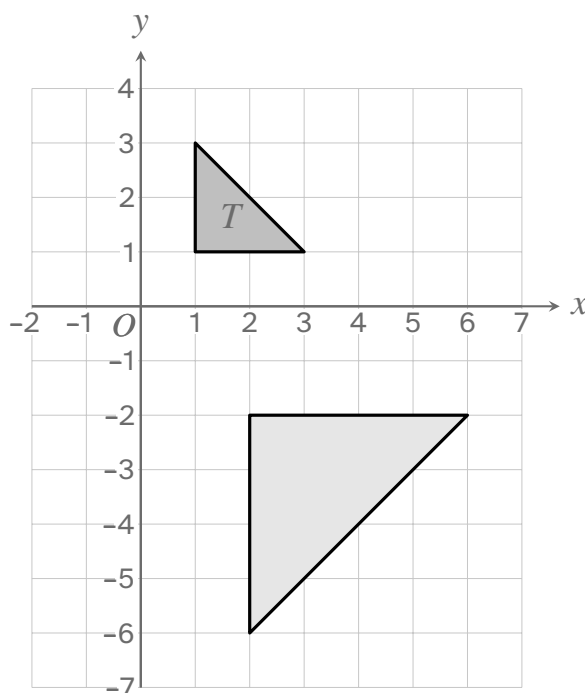
Answer: Reflection in the line $y = x$

9 (4 marks)

Triangle T is drawn on the grid.

T is enlarged by scale factor 2, centre the origin.

The image is then rotated 90° clockwise about the origin.



- (a) Use matrix multiplication to find the single matrix that represents the combined transformation.

Enlargement, scale factor 2, centre O : $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

Rotation 90° clockwise about O : $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

The enlargement happens first, so it goes on the right:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

Answer: $\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$

Tip: Here the order does not change the answer (an enlargement centre O gives the same result before or after the rotation), but always put the matrix of the transformation that happens second on the **left**.

Marks: 1 mark for both matrices correct. 1 mark for the product.

- (b) On the grid, draw the image of triangle T under the combined transformation.

Multiply each vertex by $\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$:

$$\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$$

The image has vertices $(2, -2)$, $(2, -6)$ and $(6, -2)$, shown on the grid above.

Answer: Triangle with vertices $(2, -2)$, $(2, -6)$, $(6, -2)$

Tip: Show the working for each vertex. Check the size: the image's sides should be twice as long as the sides of T .

Marks: 1 mark for two correct vertices. 1 mark for the correct triangle.
