

Matrix Multiplication

Solutions

For AQA Level 2 Certificate in Further Mathematics (8365)

Each question is shown in grey, followed by the working, one step per line. The final answer is in a box. *Marks* shows how the marks are usually given in the exam, and **Tip** points out a common mistake.

1 (1 mark)

Work out $\begin{pmatrix} 2 & -3 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix}$

Circle your answer.

Options: $\begin{pmatrix} 4 \\ 22 \end{pmatrix}$, $\begin{pmatrix} 18 \\ -13 \end{pmatrix}$, $\begin{pmatrix} -4 \\ 22 \end{pmatrix}$, $\begin{pmatrix} 10 & -6 \\ 20 & 2 \end{pmatrix}$

Multiply each **row** of the matrix by the **column**, then add.

Top row: $2 \times 5 + (-3) \times 2 = 10 - 6 = 4$

Bottom row: $4 \times 5 + 1 \times 2 = 20 + 2 = 22$

A 2×2 matrix times a 2×1 matrix gives a 2×1 matrix: $\begin{pmatrix} 4 \\ 22 \end{pmatrix}$

Answer: $\begin{pmatrix} 4 \\ 22 \end{pmatrix}$

Tip: $\begin{pmatrix} 18 \\ -13 \end{pmatrix}$ comes from multiplying down the columns instead of along the rows. Always go row of the first matrix times column of the second.

2 (4 marks)

$$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ -2 & 4 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 0 & -1 \\ 5 & 2 \end{pmatrix}$$

(a) Work out **AB**

Each entry is a row of **A** times a column of **B**, added up.

Top left (row 1, column 1): $3 \times 0 + 1 \times 5 = 5$

Top right (row 1, column 2): $3 \times (-1) + 1 \times 2 = -3 + 2 = -1$

Bottom left (row 2, column 1): $-2 \times 0 + 4 \times 5 = 20$

Bottom right (row 2, column 2): $-2 \times (-1) + 4 \times 2 = 2 + 8 = 10$

$$\mathbf{AB} = \begin{pmatrix} 5 & -1 \\ 20 & 10 \end{pmatrix}$$

Answer: $\begin{pmatrix} 5 & -1 \\ 20 & 10 \end{pmatrix}$

Tip: Row times column, then **add** the two products. Do not just multiply the entries in the same positions.

Marks: 1 mark for two correct entries. 1 mark for the fully correct matrix.

(b) Work out **BA**

Now the rows come from **B** and the columns from **A**.

Top left: $0 \times 3 + (-1) \times (-2) = 2$

Top right: $0 \times 1 + (-1) \times 4 = -4$

Bottom left: $5 \times 3 + 2 \times (-2) = 15 - 4 = 11$

Bottom right: $5 \times 1 + 2 \times 4 = 13$

$$\mathbf{BA} = \begin{pmatrix} 2 & -4 \\ 11 & 13 \end{pmatrix}$$

This is not the same as **AB**.

Answer: $\begin{pmatrix} 2 & -4 \\ 11 & 13 \end{pmatrix}$

Tip: The order matters: **AB** and **BA** are usually different. Write the first matrix named on the left.

Marks: 1 mark for two correct entries. 1 mark for the fully correct matrix.

3 (2 marks)

$$\mathbf{C} = \begin{pmatrix} 1 & -2 \\ 3 & 0 \end{pmatrix}$$

Work out $3\mathbf{C}^2$

$\mathbf{C}^2$ means $\mathbf{C} \times \mathbf{C}$

$$\mathbf{C}^2 = \begin{pmatrix} 1 & -2 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 3 & 0 \end{pmatrix}$$

Top left: $1 \times 1 + (-2) \times 3 = -5$ Top right: $1 \times (-2) + (-2) \times 0 = -2$

Bottom left: $3 \times 1 + 0 \times 3 = 3$ Bottom right: $3 \times (-2) + 0 \times 0 = -6$

$$\mathbf{C}^2 = \begin{pmatrix} -5 & -2 \\ 3 & -6 \end{pmatrix}$$

Multiply every entry by 3: $3\mathbf{C}^2 = \begin{pmatrix} -15 & -6 \\ 9 & -18 \end{pmatrix}$

Answer: $\begin{pmatrix} -15 & -6 \\ 9 & -18 \end{pmatrix}$

Tip: $\mathbf{C}^2$ is **not** found by squaring each entry, which would give $\begin{pmatrix} 1 & 4 \\ 9 & 0 \end{pmatrix}$. Multiply the matrix by itself.

Marks: 1 mark for $\mathbf{C}^2$ correct (or $3\mathbf{C}$ correct and multiplied by $\mathbf{C}$). 1 mark for the final matrix.

4 (1 mark)

$\mathbf{P}$ is a 2×2 matrix such that

$$\begin{pmatrix} 6 & -1 \\ 2 & 5 \end{pmatrix} \mathbf{P} = \begin{pmatrix} 6 & -1 \\ 2 & 5 \end{pmatrix}$$

Circle matrix $\mathbf{P}$.

Options: $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 6 & -1 \\ 2 & 5 \end{pmatrix}$

Multiplying by $\mathbf{P}$ leaves the matrix unchanged.

The matrix that leaves any matrix unchanged is the identity matrix $\mathbf{I}$.

$\mathbf{I}$ has 1s on the leading diagonal (top left to bottom right) and 0s elsewhere.

Check: $\begin{pmatrix} 6 & -1 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 6 & -1 \\ 2 & 5 \end{pmatrix} \checkmark$

Answer: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Tip: The identity matrix is not all 1s, and it is not the given matrix again. Only the leading diagonal has 1s.

5 (2 marks)

$$\mathbf{A} = \begin{pmatrix} 5 & 2 \\ -3 & 1 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 1 & -2 \\ 3 & 5 \end{pmatrix}$$

Show that $\mathbf{AB} = k\mathbf{I}$, where $\mathbf{I}$ is the identity matrix and k is an integer.
State the value of k .

$$\mathbf{AB} = \begin{pmatrix} 5 & 2 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 3 & 5 \end{pmatrix}$$

Top left: $5 \times 1 + 2 \times 3 = 11$ Top right: $5 \times (-2) + 2 \times 5 = 0$

Bottom left: $-3 \times 1 + 1 \times 3 = 0$ Bottom right: $-3 \times (-2) + 1 \times 5 = 11$

$$\mathbf{AB} = \begin{pmatrix} 11 & 0 \\ 0 & 11 \end{pmatrix} = 11 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 11\mathbf{I}$$

So $k = 11$

Answer: $\mathbf{AB} = 11\mathbf{I}$, so $k = 11$

Tip: Finish the 'show that' properly: write the product as $11\mathbf{I}$ and state $k = 11$. Stopping at $\begin{pmatrix} 11 & 0 \\ 0 & 11 \end{pmatrix}$ can lose the last mark.

Marks: 1 mark for multiplying out to get $\begin{pmatrix} 11 & 0 \\ 0 & 11 \end{pmatrix}$. 1 mark for writing it as $11\mathbf{I}$ with $k = 11$.

6 (3 marks)

$$\begin{pmatrix} p & q \\ 2q & -p \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = 11 \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Work out the values of p and q .

$$\text{Right-hand side: } 11 \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 11 \\ -22 \end{pmatrix}$$

$$\text{Top row: } 3p + 2q = 11$$

$$\text{Bottom row: } 2q \times 3 + (-p) \times 2 = -22, \text{ so } 6q - 2p = -22$$

$$\text{Divide the second equation by 2: } 3q - p = -11, \text{ so } p = 3q + 11$$

$$\text{Substitute into } 3p + 2q = 11: 3(3q + 11) + 2q = 11$$

$$9q + 33 + 2q = 11, \text{ so } 11q = -22 \text{ and } q = -2$$

$$p = 3(-2) + 11 = 5$$

$$\text{Check in the first equation: } 3(5) + 2(-2) = 11 \quad \checkmark$$

Answer: $p = 5, q = -2$

Tip: Multiply out the left-hand side before you compare: the bottom entry is $6q - 2p$, not $2q - p$. Most errors here are sign slips when solving, so check both values in both equations.

Marks: 1 mark for one correct equation. 1 mark for both equations. 1 mark for $p = 5$ and $q = -2$.

7 (3 marks)

The matrix $\mathbf{T} = \begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix}$ maps the point Q to the point $(-1, -16)$

Work out the coordinates of Q .

You must show your working.

Let Q be (x, y) . The matrix goes on the left, the point as a column on the right:

$$\begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ -16 \end{pmatrix}$$

$$\text{Top row: } 2x + 3y = -1$$

$$\text{Bottom row: } -x + 4y = -16, \text{ so } x = 4y + 16$$

$$\text{Substitute: } 2(4y + 16) + 3y = -1$$

$$8y + 32 + 3y = -1, \text{ so } 11y = -33 \text{ and } y = -3$$

$$x = 4(-3) + 16 = 4$$

$$\text{Check: } \begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} -1 \\ -16 \end{pmatrix} \checkmark$$

Answer: (4, -3)

Tip: Do not multiply the matrix by $(-1, -16)$: that would give the image of $(-1, -16)$, not the point that maps to it. Write the full equation $\mathbf{T} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ -16 \end{pmatrix}$ and solve.

Marks: 1 mark for the matrix equation (or both linear equations). 1 mark for a correct method to solve them. 1 mark for (4, -3).

8 (4 marks)

$$\mathbf{A} = \begin{pmatrix} 4 & -2 \\ b & 3 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 3 & a \\ 1 & 4 \end{pmatrix}$$

$\mathbf{AB} = k\mathbf{I}$, where $\mathbf{I}$ is the identity matrix and k is a constant.

Work out the values of a , b and k .

$$\mathbf{AB} = \begin{pmatrix} 4 & -2 \\ b & 3 \end{pmatrix} \begin{pmatrix} 3 & a \\ 1 & 4 \end{pmatrix}$$

$$\text{Top left: } 4 \times 3 + (-2) \times 1 = 10$$

$$\text{Top right: } 4a + (-2) \times 4 = 4a - 8$$

$$\text{Bottom left: } 3b + 3 \times 1 = 3b + 3$$

$$\text{Bottom right: } ab + 3 \times 4 = ab + 12$$

$$k\mathbf{I} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}, \text{ so the off-diagonal entries are 0 and the diagonal entries are both } k.$$

$$4a - 8 = 0, \text{ so } a = 2$$

$$3b + 3 = 0, \text{ so } b = -1$$

$$\text{Top left gives } k = 10$$

Check bottom right: $ab + 12 = 2 \times (-1) + 12 = 10$ ✓

Answer: $a = 2$, $b = -1$, $k = 10$

Tip: $k\mathbf{I}$ has k on the leading diagonal and 0s elsewhere. Use the 0s to find a and b , then check the bottom right entry also equals k .

Marks: 1 mark for multiplying out with at least two correct entries. 1 mark for setting the top right or bottom left entry equal to 0. 1 mark for $a = 2$ and $b = -1$. 1 mark for $k = 10$.

9 (4 marks)

$$\mathbf{A} = \begin{pmatrix} a & 2 \\ 3 & -a \end{pmatrix}$$

(a) Show that $\mathbf{A}^2 = k\mathbf{I}$, where k is an expression in terms of a .

$$\mathbf{A}^2 = \begin{pmatrix} a & 2 \\ 3 & -a \end{pmatrix} \begin{pmatrix} a & 2 \\ 3 & -a \end{pmatrix}$$

$$\text{Top left: } a \times a + 2 \times 3 = a^2 + 6$$

$$\text{Top right: } a \times 2 + 2 \times (-a) = 2a - 2a = 0$$

$$\text{Bottom left: } 3 \times a + (-a) \times 3 = 0$$

$$\text{Bottom right: } 3 \times 2 + (-a) \times (-a) = 6 + a^2$$

$$\mathbf{A}^2 = \begin{pmatrix} a^2 + 6 & 0 \\ 0 & a^2 + 6 \end{pmatrix} = (a^2 + 6)\mathbf{I}, \text{ so } k = a^2 + 6$$

Answer: $\mathbf{A}^2 = (a^2 + 6)\mathbf{I}$

Tip: Take care with $(-a) \times (-a)$: it is $+a^2$.

Marks: 1 mark for at least two correct entries of $\mathbf{A}^2$. 1 mark for the full matrix written as $(a^2 + 6)\mathbf{I}$.

(b) $\mathbf{A}^2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 20 \\ -10 \end{pmatrix}$

Work out the two possible values of a .

$$\text{From part (a), } \mathbf{A}^2 = (a^2 + 6)\mathbf{I}, \text{ so } \mathbf{A}^2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} = (a^2 + 6) \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Compare the top entries: $2(a^2 + 6) = 20$

$$a^2 + 6 = 10, \text{ so } a^2 = 4$$

$$a = 2 \text{ or } a = -2$$

(The bottom entries agree: $-1 \times 10 = -10$ ✓)

Answer: $a = 2$ or $a = -2$

Tip: $a^2 = 4$ has two solutions. Give the negative value as well as the positive one.

Marks: 1 mark for $a^2 + 6 = 10$ (or $a^2 = 4$). 1 mark for both values.
