

Index Laws and Equations with Indices

Solutions

For AQA Level 2 Certificate in Further Mathematics (8365)

Each question is shown in grey, followed by the working, one step per line. The final answer is in a box. *Marks* shows how the marks are usually given in the exam, and **Tip** points out a common mistake.

1 (1 mark)

Circle the value of $\left(\frac{8}{125}\right)^{-\frac{2}{3}}$

Options: $-\frac{4}{25}$, $\frac{4}{25}$, $\frac{5}{2}$, $\frac{25}{4}$

The negative power means flip the fraction: $\left(\frac{125}{8}\right)^{\frac{2}{3}}$

Cube root: $\sqrt[3]{\frac{125}{8}} = \frac{5}{2}$

Square: $\left(\frac{5}{2}\right)^2 = \frac{25}{4}$

Answer: $\frac{25}{4}$

2 (2 marks)

Simplify fully $(49a^{-6}b^8)^{\frac{3}{2}}$

Apply the power $\frac{3}{2}$ to each part of the bracket.

$$49^{\frac{3}{2}} = (\sqrt{49})^3 = 7^3 = 343$$

$$(a^{-6})^{\frac{3}{2}} = a^{-6 \times \frac{3}{2}} = a^{-9}$$

$$(b^8)^{\frac{3}{2}} = b^{8 \times \frac{3}{2}} = b^{12}$$

So the answer is $343a^{-9}b^{12}$, which is $\frac{343b^{12}}{a^9}$

Answer: $343a^{-9}b^{12}$ or $\frac{343b^{12}}{a^9}$

Tip: The power applies to the 49 as well: $49^{\frac{3}{2}} = 343$, not $49 \times \frac{3}{2}$. For the letters, multiply the powers.

Marks: 2 marks for a fully correct answer. 1 mark for two of 343, a^{-9} and b^{12} .

3 (4 marks)

Write each expression as a single power of x .

(a) $\frac{x^2 \times \sqrt{x}}{\sqrt[3]{x}}$

Write the roots as fractional powers: $\sqrt{x} = x^{\frac{1}{2}}$ and $\sqrt[3]{x} = x^{\frac{1}{3}}$

Multiplying: add the powers. Dividing: subtract the powers.

$$2 + \frac{1}{2} - \frac{1}{3} = \frac{12}{6} + \frac{3}{6} - \frac{2}{6} = \frac{13}{6}$$

So the expression is $x^{\frac{13}{6}}$

Answer: $x^{\frac{13}{6}}$

Tip: A square root is a power of $\frac{1}{2}$ and a cube root is a power of $\frac{1}{3}$. Use a common denominator to add and subtract them.

Marks: 1 mark for $x^{\frac{1}{2}}$ and $x^{\frac{1}{3}}$ used. 1 mark for $x^{\frac{13}{6}}$.

(b) $\left(\frac{\sqrt{x}}{x^3}\right)^{-2}$

Inside the bracket: $\frac{x^{\frac{1}{2}}}{x^3} = x^{\frac{1}{2}-3} = x^{-\frac{5}{2}}$

Power of a power, so multiply: $\left(x^{-\frac{5}{2}}\right)^{-2} = x^{-\frac{5}{2} \times (-2)}$

$$-\frac{5}{2} \times (-2) = 5, \text{ so the answer is } x^5$$

Answer: x^5

Tip: Simplify inside the bracket first. A negative times a negative is positive, so the final power is +5.

Marks: 1 mark for $x^{-\frac{5}{2}}$ (or $\frac{x^{-1}}{x^{-6}}$). 1 mark for x^5 .

4 (2 marks)

$$(t^3)^k = t^8 \div t^{2k}$$

Work out the value of k .

Left side: power of a power, so multiply: t^{3k}

Right side: dividing, so subtract: t^{8-2k}

The powers must be equal: $3k = 8 - 2k$

$$5k = 8, \text{ so } k = \frac{8}{5}$$

Answer: $k = \frac{8}{5}$

Tip: When you divide powers of the same letter, subtract the indices: $t^8 \div t^{2k} = t^{8-2k}$, not $t^{\frac{8}{2k}}$.

Marks: 1 mark for $3k = 8 - 2k$. 1 mark for $\frac{8}{5}$ (or 1.6).

5 (3 marks)

Simplify fully $\frac{15m^{\frac{3}{2}}n^{-1} - 10m^{-\frac{1}{2}}n^2}{5m^{\frac{1}{2}}n^{-3}}$

Divide each term on the top by $5m^{\frac{1}{2}}n^{-3}$ separately.

First term: $15 \div 5 = 3$, $m^{\frac{3}{2}-\frac{1}{2}} = m^1$, $n^{-1-(-3)} = n^2$, so $3mn^2$

Second term: $10 \div 5 = 2$, $m^{-\frac{1}{2}-\frac{1}{2}} = m^{-1}$, $n^{2-(-3)} = n^5$, so $2m^{-1}n^5$

Answer: $3mn^2 - 2m^{-1}n^5$, which can also be written $3mn^2 - \frac{2n^5}{m}$

Answer: $3mn^2 - 2m^{-1}n^5$ or $3mn^2 - \frac{2n^5}{m}$

Tip: Dividing by n^{-3} means subtracting -3 , which **adds** 3 to the power. Divide **both** terms on the top, and keep them separate: they are not like terms.

Marks: 1 mark for each correct term. 1 mark for the fully correct answer.

6 (5 marks)

Solve

(a) $\sqrt[3]{2x^2 - 3} = 5$

Cube both sides: $2x^2 - 3 = 125$

$$2x^2 = 128, \text{ so } x^2 = 64$$

Square root, remembering both signs: $x = 8$ or $x = -8$

Answer: $x = 8$ or $x = -8$

Tip: Cube both sides to undo the cube root (don't multiply by 3). Then $x^2 = 64$ has two solutions.

Marks: 1 mark for $2x^2 - 3 = 125$. 1 mark for both 8 and -8 .

(b) $x^{-\frac{3}{4}} = \frac{125}{64}$

The negative power means a reciprocal, so flip both sides: $x^{\frac{3}{4}} = \frac{64}{125}$

Raise both sides to the power $\frac{4}{3}$ (this undoes the power $\frac{3}{4}$): $x = \left(\frac{64}{125}\right)^{\frac{4}{3}}$

Cube root: $\sqrt[3]{\frac{64}{125}} = \frac{4}{5}$

Fourth power: $\left(\frac{4}{5}\right)^4 = \frac{256}{625}$

Answer: $x = \frac{256}{625}$

Tip: To undo a power of $\frac{3}{4}$, raise to the power $\frac{4}{3}$. Do not multiply $\frac{125}{64}$ by $\frac{4}{3}$.

Marks: 1 mark for $x^{\frac{3}{4}} = \frac{64}{125}$ or $x = \left(\frac{125}{64}\right)^{-\frac{4}{3}}$. 1 mark for $\left(\frac{4}{5}\right)^4$. 1 mark for $\frac{256}{625}$.

7 (3 marks)

Solve $\frac{3^{x+5}}{9^x} = 27^{2x}$

Write everything as a power of 3:

$$9^x = (3^2)^x = 3^{2x} \text{ and } 27^{2x} = (3^3)^{2x} = 3^{6x}$$

Left side: dividing, so subtract the powers: $3^{x+5-2x} = 3^{5-x}$

$$\text{So } 3^{5-x} = 3^{6x}$$

The powers must be equal: $5 - x = 6x$

$$7x = 5, \text{ so } x = \frac{5}{7}$$

Answer: $x = \frac{5}{7}$

Tip: When you divide powers of the same base, subtract the indices. Bracket the power you are taking away: $(x + 5) - 2x$.

Marks: 1 mark for 3^{2x} or 3^{6x} . 1 mark for $5 - x = 6x$ (or $x + 5 - 2x = 6x$). 1 mark for $\frac{5}{7}$.

8 (3 marks)

The curves $y = \left(\frac{1}{2}\right)^x$ and $y = 4^{x+3}$ intersect at the point P .

Work out the coordinates of P .

At P the y values are equal: $\left(\frac{1}{2}\right)^x = 4^{x+3}$

Write both sides as powers of 2:

$$\left(\frac{1}{2}\right)^x = (2^{-1})^x = 2^{-x} \text{ and } 4^{x+3} = (2^2)^{x+3} = 2^{2x+6}$$

Equate the powers: $-x = 2x + 6$, so $x = -2$

$$y = \left(\frac{1}{2}\right)^{-2} = 2^2 = 4$$

(Check: $4^{-2+3} = 4^1 = 4$ ✓)

Answer: $P(-2, 4)$

Tip: $\left(\frac{1}{2}\right)^{-2} = 2^2 = 4$, not $\frac{1}{4}$ or -4 . A negative power flips the fraction. On a calculator, put $\frac{1}{2}$ in brackets.

Marks: 1 mark for 2^{-x} or 2^{2x+6} . 1 mark for $x = -2$. 1 mark for $(-2, 4)$.

9 (4 marks)

Solve $2^{x^2} \times 4^x = 8^{x+4}$

You must show your working.

Write each side as a power of 2:

$$4^x = (2^2)^x = 2^{2x} \text{ and } 8^{x+4} = (2^3)^{x+4} = 2^{3x+12}$$

Left side: multiplying, so add the powers: 2^{x^2+2x}

$$\text{So } 2^{x^2+2x} = 2^{3x+12}$$

Equate the powers: $x^2 + 2x = 3x + 12$

$$x^2 - x - 12 = 0$$

$$(x - 4)(x + 3) = 0$$

$$x = 4 \text{ or } x = -3$$

Answer: $x = 4$ or $x = -3$

Tip: Multiply every term in the bracket: $3(x + 4) = 3x + 12$, not $3x + 4$. Once the bases match, the powers must be equal, which gives a quadratic.

Marks: 1 mark for 2^{2x} or 2^{3x+12} . 1 mark for $x^2 + 2x = 3x + 12$. 1 mark for factorising $x^2 - x - 12$ (or using the formula). 1 mark for both answers.

10 (3 marks)

Solve $3^{x+2} + 3^x = 30\sqrt{3}$

Do not use trial and improvement.

You must show your working.

$$3^{x+2} = 3^x \times 3^2 = 9 \times 3^x$$

$$\text{So the left side is } 9 \times 3^x + 3^x = 10 \times 3^x$$

$$10 \times 3^x = 30\sqrt{3}, \text{ so } 3^x = 3\sqrt{3}$$

$$\text{Write the right side as a power of 3: } 3\sqrt{3} = 3^1 \times 3^{\frac{1}{2}} = 3^{\frac{3}{2}}$$

$$3^x = 3^{\frac{3}{2}}, \text{ so } x = \frac{3}{2}$$

Answer: $x = \frac{3}{2}$

Tip: $3^{x+2} + 3^x$ is not 3^{2x+2} : you only add the powers when you **multiply**. Write 3^{x+2} as 9×3^x and collect the 3^x terms.

Marks: 1 mark for $3^{x+2} = 9 \times 3^x$ or $3^x(9 + 1)$. 1 mark for $3^x = 3\sqrt{3}$. 1 mark for $\frac{3}{2}$ (or 1.5).

11 (4 marks)

Solve $\left(x^{\frac{3}{2}} + 4x^{\frac{1}{2}}\right)\left(x^{\frac{3}{2}} - x^{\frac{1}{2}}\right) = 6x(x + 1)$ for $x \geq 0$

You must show your working.

Expand the brackets: multiply each term in the first bracket by each term in the second.

$$x^{\frac{3}{2}} \times x^{\frac{3}{2}} = x^3 \quad \text{and} \quad x^{\frac{3}{2}} \times (-x^{\frac{1}{2}}) = -x^2$$

$$4x^{\frac{1}{2}} \times x^{\frac{3}{2}} = 4x^2 \quad \text{and} \quad 4x^{\frac{1}{2}} \times (-x^{\frac{1}{2}}) = -4x$$

So the left side is $x^3 + 3x^2 - 4x$

Right side: $6x(x + 1) = 6x^2 + 6x$

$$x^3 + 3x^2 - 4x = 6x^2 + 6x$$

Rearrange: $x^3 - 3x^2 - 10x = 0$

Factorise (do not divide by x): $x(x^2 - 3x - 10) = 0$

$$x(x - 5)(x + 2) = 0$$

$$x = 0, x = 5 \text{ or } x = -2$$

$x \geq 0$, so reject $x = -2$

Answer: $x = 0$ or $x = 5$

Tip: Multiplying powers of x means adding the indices: $x^{\frac{1}{2}} \times x^{\frac{3}{2}} = x^2$, not $x^{\frac{3}{4}}$. Factorise out x rather than dividing by it, or you lose the solution $x = 0$.

Marks: 1 mark for expanding the left side to $x^3 + 3x^2 - 4x$. 1 mark for $x^3 - 3x^2 - 10x = 0$. 1 mark for $x(x - 5)(x + 2)$. 1 mark for $x = 0$ and $x = 5$ only.

12 (4 marks)

Here is an equation, where $x > 0$

$$10x^{\frac{1}{2}} - 3x^{\frac{3}{2}} = 5x^{-\frac{1}{2}}$$

(a) Show that the equation can be written as $3x^2 - 10x + 5 = 0$

Multiply every term by $x^{\frac{1}{2}}$ (add $\frac{1}{2}$ to each power):

$$10x^{\frac{1}{2}} \times x^{\frac{1}{2}} = 10x$$

$$3x^{\frac{3}{2}} \times x^{\frac{1}{2}} = 3x^2$$

$$5x^{-\frac{1}{2}} \times x^{\frac{1}{2}} = 5x^0 = 5$$

$$\text{So } 10x - 3x^2 = 5$$

Rearrange: $3x^2 - 10x + 5 = 0$ ✓

Answer: Multiplying by $x^{\frac{1}{2}}$ gives $10x - 3x^2 = 5$, so $3x^2 - 10x + 5 = 0$

Tip: $x^{-\frac{1}{2}} \times x^{\frac{1}{2}} = x^0 = 1$, so the last term becomes just 5.

Marks: 1 mark for multiplying through by $x^{\frac{1}{2}}$ with at least two terms correct. 1 mark for reaching $3x^2 - 10x + 5 = 0$ with all steps shown.

- (b) Hence solve the equation.
Give your answers to 3 significant figures.

Use the quadratic formula with $a = 3$, $b = -10$, $c = 5$:

$$x = \frac{10 \pm \sqrt{(-10)^2 - 4 \times 3 \times 5}}{2 \times 3} = \frac{10 \pm \sqrt{40}}{6}$$

$$x = \frac{10 + \sqrt{40}}{6} = 2.7207 \dots \text{ or } x = \frac{10 - \sqrt{40}}{6} = 0.6125 \dots$$

Both are positive, so both are solutions (check: $x^{\frac{1}{2}}$ needs $x > 0$).

Answer: $x = 2.72$ or $x = 0.613$

Tip: Write the substitution in full, with $(-10)^2$ in brackets. Give both answers to 3 significant figures: 0.613, not 0.61.

Marks: 1 mark for a correct substitution into the formula. 1 mark for both answers to 3 s.f.