

Geometric Proof

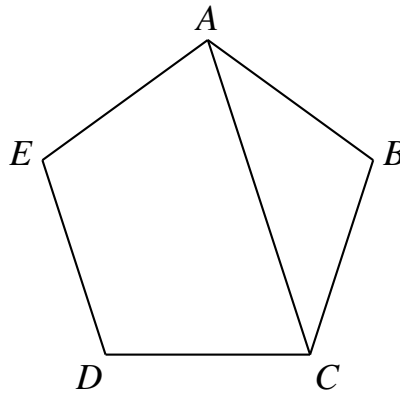
Solutions

For AQA Level 2 Certificate in Further Mathematics (8365)

Each question is shown in grey, followed by the working, one step per line. The final answer is in a box. *Marks* shows how the marks are usually given in the exam, and **Tip** points out a common mistake.

1 (3 marks)

ABCDE is a regular pentagon.



Not drawn accurately

Prove that *AC* is parallel to *ED*.

Give a reason for each stage of your working.

The interior angles of a pentagon add up to $(5 - 2) \times 180 = 540^\circ$.

The pentagon is regular, so each interior angle is $540 \div 5 = 108^\circ$.

$AB = BC$ (the sides of a regular pentagon are equal), so triangle *ABC* is isosceles.

Angle $BCA = (180 - 108) \div 2 = 36^\circ$

(Base angles of an isosceles triangle are equal; angles in a triangle add up to 180° .)

Angle $ACD = 108 - 36 = 72^\circ$

Angle $ACD + \text{angle } CDE = 72 + 108 = 180^\circ$

These are co-interior angles between *AC* and *ED*, and they add up to 180° .

So AC is parallel to ED .

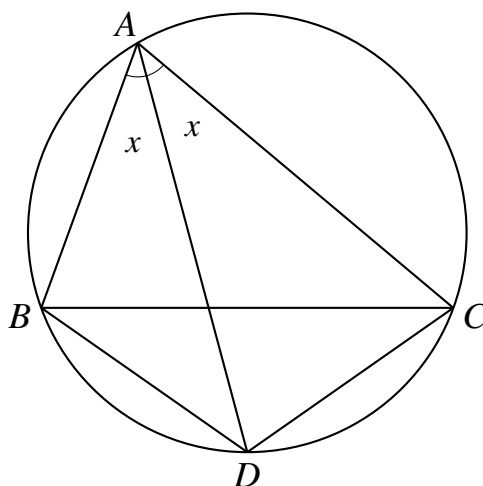
Answer: Angle ACD + angle $CDE = 72^\circ + 108^\circ = 180^\circ$, so the co-interior angles add up to 180° and AC is parallel to ED .

Tip: Finish with a statement: “co-interior angles add up to 180° , so AC is parallel to ED ”.
Showing the numbers without the conclusion loses the last mark.

Marks: 1 mark for the interior angle 108° . 1 mark for angle $BCA = 36^\circ$ or angle $ACD = 72^\circ$. 1 mark for a complete proof with the co-interior angle reason.

2 (3 marks)

A, B, C and D are points on a circle.



Not drawn accurately

Angle BAD = angle CAD

Prove that triangle BCD is isosceles.

Give a reason for each stage of your working.

Let angle BAD = angle CAD = x .

Angle DBC = angle DAC = x

(Angles in the same segment are equal: both stand on the chord DC .)

Angle DCB = angle DAB = x

(Angles in the same segment are equal: both stand on the chord DB .)

So angle DBC = angle DCB .

Triangle BCD has two equal angles, so it is isosceles (with $DB = DC$).

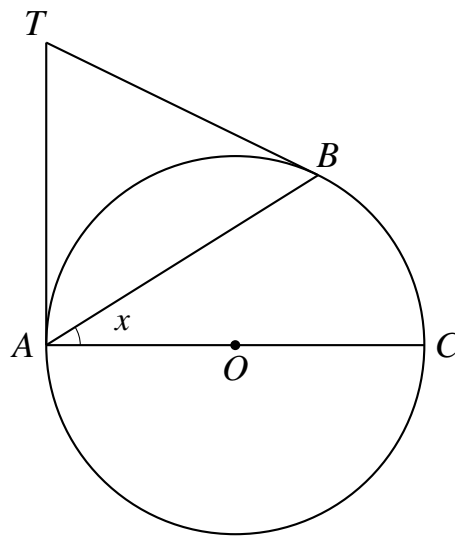
Answer: Angle $DBC = \text{angle } DCB = x$, so triangle BCD is isosceles.

Tip: For each pair, say which chord both angles stand on. Name angles with three letters (angle DBC , not angle B): there are several angles at B .

Marks: 1 mark for one angle of triangle BCD equal to x with the reason. 1 mark for the second angle with the reason. 1 mark for the conclusion.

3 (4 marks)

A , B and C are points on a circle, centre O .



Not drawn accurately

AC is a diameter.

TA and TB are tangents to the circle.

Angle $BAC = x$

Prove that angle $ATB = 2x$

Give a reason for each stage of your working.

Angle $TAC = 90^\circ$

(A tangent is perpendicular to the radius; OA is part of the diameter AC .)

So angle $TAB = 90 - x$

$TA = TB$ (tangents from a point to a circle are equal), so triangle TAB is isosceles.

Angle $TBA = \text{angle } TAB = 90 - x$

(Base angles of an isosceles triangle are equal.)

$$\text{Angle } ATB = 180 - 2(90 - x)$$

(Angles in a triangle add up to 180° .)

$$= 180 - 180 + 2x = 2x$$

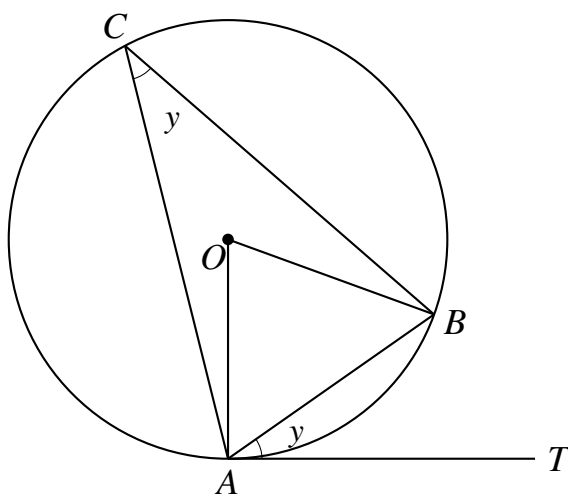
Answer: Angle $ATB = 180 - 2(90 - x) = 2x$

Tip: Bracket the expression: $180 - 2(90 - x) = 180 - 180 + 2x$. Writing $180 - 2 \times 90 - x$ gives the wrong sign. Another route: angle $AOB = 180 - 2x$ (isosceles triangle AOB), then angles in quadrilateral $OATB$.

Marks: 1 mark for angle $TAC = 90^\circ$ with the reason. 1 mark for angle $TAB = 90 - x$. 1 mark for angle $TBA = 90 - x$ with the reason. 1 mark for a complete proof.

4 (3 marks)

A , B and C are points on a circle, centre O .



Not drawn accurately

T is a point such that angle $TAB = \text{angle } ACB = y$

Prove that TA is a tangent to the circle.

Give a reason for each stage of your working.

$$\text{Angle } AOB = 2y$$

(The angle at the centre is twice the angle at the circumference.)

$OA = OB$ (radii), so triangle OAB is isosceles.

$$\text{Angle } OAB = (180 - 2y) \div 2 = 90 - y$$

(Base angles of an isosceles triangle are equal; angles in a triangle add up to 180° .)

$$\text{Angle } OAT = \text{angle } OAB + \text{angle } BAT = (90 - y) + y = 90^\circ$$

So TA is perpendicular to the radius OA at A .

A line perpendicular to a radius at the point where it meets the circle is a tangent, so TA is a tangent.

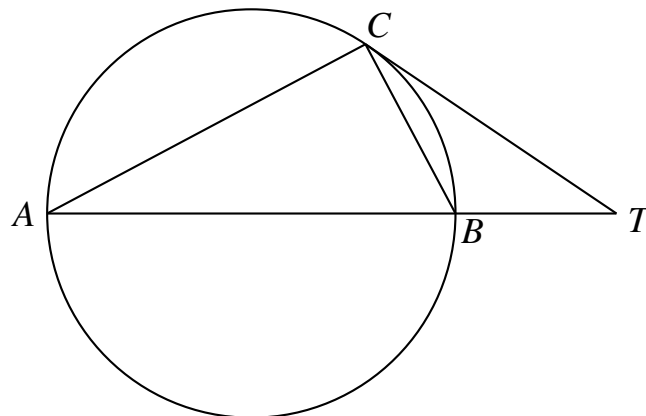
Answer: Angle $OAT = (90 - y) + y = 90^\circ$, so TA is perpendicular to the radius OA and is a tangent.

Tip: You cannot use the alternate segment theorem here: it assumes TA is already a tangent, which is what you are proving. Show the angle between TA and the radius is 90° .

Marks: 1 mark for angle $AOB = 2y$ with the reason. 1 mark for angle $OAB = 90 - y$. 1 mark for angle $OAT = 90^\circ$ with the conclusion.

5 (4 marks)

A , B and C are points on a circle.



Not drawn accurately

The tangent to the circle at C meets the line AB extended at T .

$$\text{Angle } BCT = y \quad \text{Angle } ATC = 90^\circ - 2y$$

Prove that AB is a diameter of the circle.

Give a reason for each stage of your working.

$$\text{Angle } CAB = \text{angle } BCT = y$$

(Alternate segment theorem.)

In triangle ACT : angle $ACT = 180 - y - (90 - 2y)$

(Angles in a triangle add up to 180° .)

$$= 180 - y - 90 + 2y = 90 + y$$

$$\text{Angle } ACB = \text{angle } ACT - \text{angle } BCT = (90 + y) - y = 90^\circ$$

AB subtends a right angle at C on the circle.

The angle in a semicircle is 90° , so AB is a diameter.

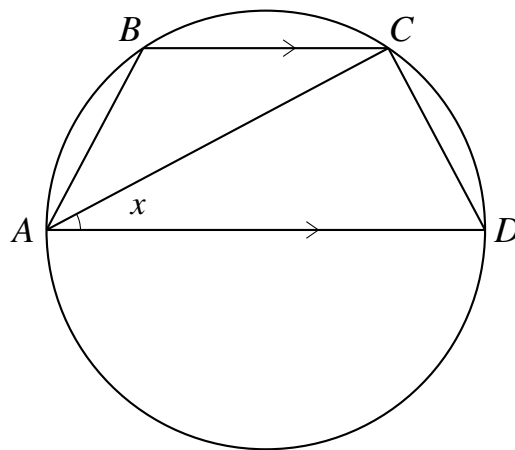
Answer: Angle $ACB = (90 + y) - y = 90^\circ$, so AB is a diameter.

Tip: Bracket the expression you subtract: $180 - y - (90 - 2y)$. Without brackets the $-2y$ does not become $+2y$. Another route: angle $ABC = y + (90 - 2y) = 90 - y$ (exterior angle of triangle BCT), then the angles in triangle ABC .

Marks: 1 mark for angle $CAB = y$ with the reason. 1 mark for angle $ACT = 90 + y$ (or angle $ABC = 90 - y$). 1 mark for angle $ACB = 90^\circ$. 1 mark for the conclusion with reasons.

6 (4 marks)

A, B, C and D are points on a circle.



Not drawn accurately

AD is a diameter of the circle.

BC is parallel to AD .

Angle $CAD = x$

Prove that angle $BAC = 90^\circ - 2x$

Give a reason for each stage of your working.

Angle $ACD = 90^\circ$

(The angle in a semicircle is 90° .)

$$\text{Angle } ADC = 180 - 90 - x = 90 - x$$

(Angles in a triangle add up to 180° .)

$$\text{Angle } ABC = 180 - (90 - x) = 90 + x$$

(Opposite angles of a cyclic quadrilateral add up to 180° .)

$$\text{Angle } BCA = \text{angle } CAD = x$$

(Alternate angles are equal, because BC is parallel to AD .)

$$\text{In triangle } ABC: \text{ angle } BAC = 180 - (90 + x) - x$$

(Angles in a triangle add up to 180° .)

$$= 180 - 90 - x - x = 90 - 2x$$

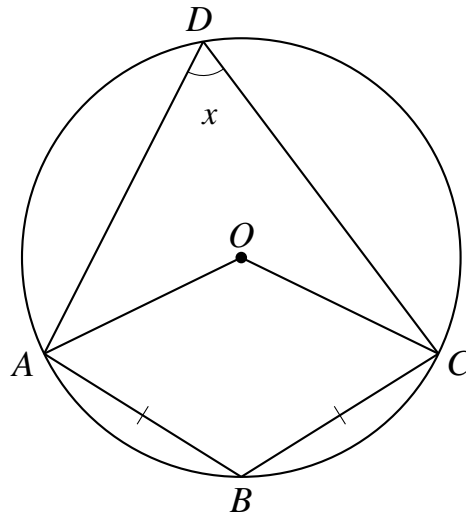
Answer: Angle $BAC = 180 - (90 + x) - x = 90^\circ - 2x$

Tip: Opposite angles of a cyclic quadrilateral add up to 180° : they are not equal. Bracket what you subtract, $180 - (90 + x) - x$, so the signs come out right. Another route: join BD ; angle $BDA = \text{angle } BCA = x$ (same segment), so angle $BDC = (90 - x) - x$, and angle $BAC = \text{angle } BDC$ (same segment).

Marks: 1 mark for angle $ACD = 90^\circ$ or angle $ADC = 90 - x$ with the reason. 1 mark for angle $ABC = 90 + x$ with the reason. 1 mark for angle $BCA = x$ with the reason. 1 mark for a complete proof.

7 (5 marks)

A, B, C and D are points on a circle, centre O .



Not drawn accurately

$AB = BC$ Angle $ADC = x$
Prove that angle $OAB = 90^\circ - \frac{x}{2}$

Give a reason for each stage of your working.

$$\text{Angle } AOC = 2x$$

(The angle at the centre is twice the angle at the circumference.)

$$OA = OC \text{ (radii), so angle } OAC = (180 - 2x) \div 2 = 90 - x$$

(Base angles of an isosceles triangle are equal; angles in a triangle add up to 180° .)

$$\text{Angle } ABC = 180 - x$$

(Opposite angles of a cyclic quadrilateral add up to 180° .)

$$AB = BC, \text{ so angle } BAC = \text{angle } BCA = \frac{180 - (180 - x)}{2} = \frac{x}{2}$$

(Base angles of an isosceles triangle are equal; angles in a triangle add up to 180° .)

$$\text{Angle } OAB = \text{angle } OAC + \text{angle } CAB = (90 - x) + \frac{x}{2} = 90 - \frac{x}{2}$$

$$\textbf{Answer:} \text{ Angle } OAB = (90 - x) + \frac{x}{2} = 90^\circ - \frac{x}{2}$$

Tip: Join AC first and mark each angle in terms of x on the diagram. Another route: join OB ; triangles OAB and OCB are congruent (SSS), so angle $AOB = x$ and angle $OAB = (180 - x) \div 2 = 90 - \frac{x}{2}$.

Marks: 1 mark for angle $AOC = 2x$ with the reason. 1 mark for angle $OAC = 90 - x$. 1 mark for angle $ABC = 180 - x$ with the reason. 1 mark for angle $BAC = \frac{x}{2}$. 1 mark for a complete proof with all reasons.
