

Factor Theorem

Solutions

For AQA Level 2 Certificate in Further Mathematics (8365)

Each question is shown in grey, followed by the working, one step per line. The final answer is in a box. *Marks* shows how the marks are usually given in the exam, and **Tip** points out a common mistake.

1 (1 mark)

You want to use the factor theorem to test whether $(3x + 2)$ is a factor of a polynomial $f(x)$

Which value of x should you substitute into $f(x)$?

Circle your answer.

Options: $x = -\frac{3}{2}$, $x = -\frac{2}{3}$, $x = \frac{2}{3}$, $x = 2$

Substitute the value of x that makes the bracket equal to 0.

$$3x + 2 = 0$$

$$3x = -2$$

$$x = -\frac{2}{3}$$

If $f\left(-\frac{2}{3}\right) = 0$, then $(3x + 2)$ is a factor.

Answer: $x = -\frac{2}{3}$

Tip: Set the bracket equal to 0 and solve. Using $x = \frac{2}{3}$ (the wrong sign) or $x = -\frac{3}{2}$ (upside down) are the usual mistakes.

2 (2 marks)

$$p(x) = x^4 + 3x^3 + x^2 + 4$$

Use the factor theorem to show that $(x + 2)$ is a factor of $p(x)$

$(x + 2)$ is a factor if $p(-2) = 0$

$$p(-2) = (-2)^4 + 3(-2)^3 + (-2)^2 + 4$$

$$= 16 + 3 \times (-8) + 4 + 4$$

$$= 16 - 24 + 4 + 4$$

$$= 0$$

$p(-2) = 0$, so $(x + 2)$ is a factor of $p(x)$.

Answer: $p(-2) = 0$, so $(x + 2)$ is a factor

Tip: Put brackets round the -2 every time: $(-2)^4 = +16$ but $(-2)^3 = -8$. Always finish with a sentence saying what $p(-2) = 0$ tells you.

Marks: 1 mark for substituting $x = -2$. 1 mark for showing it equals 0 **and** saying this means $(x + 2)$ is a factor.

3 (3 marks)

$$(x - 5)(x^2 + ax + b) \equiv x^3 - 2x^2 - 13x - 10$$

(a) Work out the values of a and b .

Expand the left-hand side in your head, one power of x at a time.

Constant terms: $-5 \times b = -10$, so $b = 2$

x^2 terms: $x \times ax$ gives ax^2 , and $-5 \times x^2$ gives $-5x^2$

So $a - 5 = -2$, which gives $a = 3$

Check the x terms: $x \times b + (-5) \times ax = 2x - 15x = -13x$ ✓

Answer: $a = 3$, $b = 2$

Tip: Compare the constant terms first: they give b straight away. Then use the x^2 terms. The x terms are a free check.

Marks: 1 mark for each value.

(b) Hence factorise $x^3 - 2x^2 - 13x - 10$ fully.

From part (a): $x^3 - 2x^2 - 13x - 10 = (x - 5)(x^2 + 3x + 2)$

Factorise the quadratic: $x^2 + 3x + 2 = (x + 1)(x + 2)$

$$\text{So } x^3 - 2x^2 - 13x - 10 = (x - 5)(x + 1)(x + 2)$$

Answer: $(x - 5)(x + 1)(x + 2)$

4 (5 marks)

$$f(x) = 3x^3 + 4x^2 - 13x + 6$$

- (a) By working out $f\left(\frac{2}{3}\right)$, show that $(3x - 2)$ is a factor of $f(x)$

$$3x - 2 = 0 \text{ when } x = \frac{2}{3}$$

$$f\left(\frac{2}{3}\right) = 3\left(\frac{2}{3}\right)^3 + 4\left(\frac{2}{3}\right)^2 - 13\left(\frac{2}{3}\right) + 6$$

$$= 3 \times \frac{8}{27} + 4 \times \frac{4}{9} - \frac{26}{3} + 6$$

$$= \frac{8}{9} + \frac{16}{9} - \frac{78}{9} + \frac{54}{9}$$

$$= 0$$

$$f\left(\frac{2}{3}\right) = 0, \text{ so } (3x - 2) \text{ is a factor of } f(x).$$

Answer: $f\left(\frac{2}{3}\right) = 0$, so $(3x - 2)$ is a factor

Tip: Show every term of the substitution and the arithmetic, then finish with a sentence: " $f\left(\frac{2}{3}\right) = 0$, so $(3x - 2)$ is a factor". When a question says use the factor theorem, dividing $f(x)$ by the bracket instead scores nothing.

Marks: 1 mark for substituting $x = \frac{2}{3}$. 1 mark for showing it equals 0 with a conclusion.

- (b) Factorise $f(x)$ fully and hence solve $f(x) = 0$

$$\text{Write } f(x) = (3x - 2)(x^2 + bx + c)$$

$$\text{Constant terms: } -2 \times c = 6, \text{ so } c = -3$$

$$x^2 \text{ terms: } 3b - 2 = 4, \text{ so } b = 2$$

$$\text{Check the } x \text{ terms: } 3c - 2b = -9 - 4 = -13 \quad \checkmark$$

$$\text{So } f(x) = (3x - 2)(x^2 + 2x - 3)$$

$$= (3x - 2)(x + 3)(x - 1)$$

$$f(x) = 0 \text{ when } 3x - 2 = 0, \ x + 3 = 0 \text{ or } x - 1 = 0$$

Answer: $x = \frac{2}{3}, \ x = -3, \ x = 1$

Tip: Don't stop at the brackets: "solve" means give the values of x . And remember the solution from the factor you were given, $x = \frac{2}{3}$.

Marks: 1 mark for the quadratic factor $x^2 + 2x - 3$. 1 mark for $(3x - 2)(x + 3)(x - 1)$. 1 mark for all three solutions.

5 (3 marks)

$(2x + 1)$ is a factor of $4x^4 + ax^3 - 3x + 1$

Work out the value of a .

$$2x + 1 = 0 \text{ when } x = -\frac{1}{2}$$

By the factor theorem, substituting $x = -\frac{1}{2}$ gives 0:

$$4\left(-\frac{1}{2}\right)^4 + a\left(-\frac{1}{2}\right)^3 - 3\left(-\frac{1}{2}\right) + 1 = 0$$

$$4 \times \frac{1}{16} - \frac{a}{8} + \frac{3}{2} + 1 = 0$$

$$\frac{1}{4} + \frac{3}{2} + 1 - \frac{a}{8} = 0$$

$$\frac{11}{4} = \frac{a}{8}$$

$$a = 8 \times \frac{11}{4} = 22$$

Answer: $a = 22$

Tip: Spot that this is a factor theorem question: substitute $x = -\frac{1}{2}$ and set the result equal to 0. A negative number to an odd power is negative: $\left(-\frac{1}{2}\right)^3 = -\frac{1}{8}$.

Marks: 1 mark for substituting $x = -\frac{1}{2}$. 1 mark for setting the result equal to 0. 1 mark for $a = 22$.

6 (5 marks)

$$f(x) = x^3 - 5x^2 + x + 3$$

- (a) Use the factor theorem to find a linear factor of $f(x)$

If $(x - k)$ is a factor, then k must divide the constant term, 3.

So try $x = 1, -1, 3$ and -3 .

$$f(1) = 1 - 5 + 1 + 3 = 0$$

$f(1) = 0$, so $(x - 1)$ is a factor of $f(x)$.

Answer: $(x - 1)$

Tip: The factor is $(x - 1)$, not $(x + 1)$: $f(1) = 0$ means $x = 1$ is a solution, and the factor is $(x - 1)$.

Marks: 1 mark for showing $f(1) = 0$. 1 mark for $(x - 1)$.

- (b) Hence solve $f(x) = 0$

Give the two non-integer solutions in the form $p \pm \sqrt{q}$, where p and q are integers.

$$\text{Write } f(x) = (x - 1)(x^2 + bx + c)$$

$$\text{Constant terms: } -1 \times c = 3, \text{ so } c = -3$$

$$x^2 \text{ terms: } b - 1 = -5, \text{ so } b = -4$$

$$\text{Check the } x \text{ terms: } c - b = -3 + 4 = 1 \quad \checkmark$$

$$\text{So } f(x) = (x - 1)(x^2 - 4x - 3)$$

$x^2 - 4x - 3$ does not factorise, so use the quadratic formula:

$$x = \frac{4 \pm \sqrt{(-4)^2 - 4 \times 1 \times (-3)}}{2} = \frac{4 \pm \sqrt{28}}{2}$$

$$\sqrt{28} = \sqrt{4 \times 7} = 2\sqrt{7}$$

$$x = \frac{4 \pm 2\sqrt{7}}{2} = 2 \pm \sqrt{7}$$

Answer: $x = 1$, $x = 2 + \sqrt{7}$, $x = 2 - \sqrt{7}$

Tip: Leave the answers as surds; rounded decimals such as 4.65 lose the mark. Don't forget the solution $x = 1$ from the linear factor.

Marks: 1 mark for the quadratic factor $x^2 - 4x - 3$. 1 mark for a correct method to solve it (formula or completing the square). 1 mark for $2 \pm \sqrt{7}$ **and** $x = 1$.

7 (3 marks)

$(x - 2)$ is a factor of $g(x) = 4x^3 - 4x^2 - 7x - 2$

Kofi says,

“ $g(x) = 0$ is a cubic equation, so it must have three different solutions.”

Show that Kofi is wrong.

Write $g(x) = (x - 2)(4x^2 + bx + c)$

Constant terms: $-2 \times c = -2$, so $c = 1$

x^2 terms: $b - 8 = -4$, so $b = 4$

Check the x terms: $c - 2b = 1 - 8 = -7$ ✓

So $g(x) = (x - 2)(4x^2 + 4x + 1)$

$4x^2 + 4x + 1 = (2x + 1)^2$, so $g(x) = (x - 2)(2x + 1)^2$

$g(x) = 0$ when $x = 2$ or $x = -\frac{1}{2}$

$(2x + 1)$ is a repeated factor, so it gives only one solution.

$g(x) = 0$ has only two different solutions, so Kofi is wrong.

Answer: $g(x) = (x - 2)(2x + 1)^2$, so the only solutions are $x = 2$ and $x = -\frac{1}{2}$: two, not three

Tip: Show the full factorisation and say that the repeated factor gives the same solution twice. Solutions from a calculator alone are not enough.

Marks: 1 mark for the quadratic factor $4x^2 + 4x + 1$. 1 mark for $(x - 2)(2x + 1)^2$. 1 mark for the two solutions with a conclusion.

8 (6 marks)

$f(x) = 2x^3 + ax^2 + bx + 12$, where a and b are constants.

$(x + 4)$ is a factor of $f(x)$

$f(1) = -10$

(a) Work out the values of a and b .

$(x + 4)$ is a factor, so $f(-4) = 0$:

$$2(-4)^3 + a(-4)^2 + b(-4) + 12 = 0$$

$$-128 + 16a - 4b + 12 = 0$$

$$16a - 4b = 116, \text{ which simplifies to } 4a - b = 29 \quad (1)$$

$f(1) = -10$:

$$2 + a + b + 12 = -10$$

$$a + b = -24 \quad (2)$$

Add (1) and (2): $5a = 5$, so $a = 1$

Substitute into (2): $1 + b = -24$, so $b = -25$

Check in (1): $4 \times 1 - (-25) = 29 \quad \checkmark$

Answer: $a = 1$, $b = -25$

Tip: Use each piece of information to make one equation, then solve them as simultaneous equations. Put brackets round -4 : $2(-4)^3 = -128$.

Marks: 1 mark for $f(-4) = 0$ as an equation. 1 mark for $f(1) = -10$ as an equation. 1 mark for solving them. 1 mark for both values.

(b) Factorise $f(x)$ fully.

$$f(x) = 2x^3 + x^2 - 25x + 12$$

$$\text{Write } f(x) = (x + 4)(2x^2 + px + q)$$

$$\text{Constant terms: } 4q = 12, \text{ so } q = 3$$

$$x^2 \text{ terms: } p + 8 = 1, \text{ so } p = -7$$

$$\text{Check the } x \text{ terms: } q + 4p = 3 - 28 = -25 \quad \checkmark$$

$$\text{So } f(x) = (x + 4)(2x^2 - 7x + 3)$$

$$2x^2 - 7x + 3 = (2x - 1)(x - 3)$$

$$\textbf{Answer: } (x + 4)(2x - 1)(x - 3)$$

Tip: Check your answer by expanding the three brackets: you should get back to $2x^3 + x^2 - 25x + 12$.

Marks: 1 mark for the quadratic factor $2x^2 - 7x + 3$. 1 mark for the full factorisation.
