

Differentiation

Solutions

For AQA Level 2 Certificate in Further Mathematics (8365)

Each question is shown in grey, followed by the working, one step per line. The final answer is in a box. *Marks* shows how the marks are usually given in the exam, and **Tip** points out a common mistake.

1 (4 marks)

Work out $\frac{dy}{dx}$ for each curve.

(a) $y = 4x^5 - 3x^2 + 8x - 9$

Differentiate each term: multiply by the power, then take 1 off the power.

$$4x^5 \rightarrow 5 \times 4x^4 = 20x^4$$

$$-3x^2 \rightarrow 2 \times (-3)x^1 = -6x$$

$$8x \rightarrow 8$$

-9 is a constant, so it differentiates to 0.

$$\frac{dy}{dx} = 20x^4 - 6x + 8$$

Answer: $\frac{dy}{dx} = 20x^4 - 6x + 8$

Marks: 1 mark for two terms correct. 1 mark for all three terms correct (and no constant term).

(b) $y = \frac{x^3}{6} - \frac{5}{2x^4}$

First write each term as a number times a power of x :

$$\frac{x^3}{6} = \frac{1}{6}x^3$$

$$\frac{5}{2x^4} = \frac{5}{2} \times \frac{1}{x^4} = \frac{5}{2}x^{-4}$$

$$\text{So } y = \frac{1}{6}x^3 - \frac{5}{2}x^{-4}$$

$$\frac{1}{6}x^3 \rightarrow 3 \times \frac{1}{6}x^2 = \frac{1}{2}x^2$$

$$-\frac{5}{2}x^{-4} \rightarrow (-4) \times \left(-\frac{5}{2}\right)x^{-5} = 10x^{-5}$$

$$\frac{dy}{dx} = \frac{1}{2}x^2 + 10x^{-5}$$

$$\text{Answer: } \frac{dy}{dx} = \frac{1}{2}x^2 + 10x^{-5} \text{ (or } \frac{x^2}{2} + \frac{10}{x^5})$$

Tip: $\frac{5}{2x^4}$ is $\frac{5}{2}x^{-4}$, not $10x^{-4}$: only the x^4 moves up as x^{-4} , the 2 stays underneath the 5.
For a negative power, taking 1 off makes it **more** negative: $x^{-4} \rightarrow x^{-5}$.

Marks: 1 mark for $-\frac{5}{2}x^{-4}$ (or for one term correct). 1 mark for the fully correct answer.

2 (3 marks)

$$y = \frac{4}{x}(x^3 - 2x + 3)$$

Work out $\frac{dy}{dx}$

You cannot differentiate a product directly, so expand the bracket first.

$\frac{4}{x} = 4x^{-1}$. Multiply each term by $4x^{-1}$ and add the powers:

$$4x^{-1} \times x^3 = 4x^2$$

$$4x^{-1} \times (-2x) = -8x^0 = -8$$

$$4x^{-1} \times 3 = 12x^{-1}$$

$$\text{So } y = 4x^2 - 8 + 12x^{-1}$$

Now differentiate each term: $\frac{dy}{dx} = 8x - 12x^{-2}$

Answer: $\frac{dy}{dx} = 8x - 12x^{-2}$ (or $8x - \frac{12}{x^2}$)

Tip: Expand first, then differentiate. Differentiating $\frac{4}{x}$ and the bracket separately and multiplying the results is wrong. When you multiply powers of x , **add** the powers: $x^{-1} \times x^3 = x^2$.

Marks: 1 mark for expanding to $4x^2 - 8 + 12x^{-1}$ (two terms correct is enough for this mark). 1 mark for differentiating at least two terms correctly. 1 mark for the fully correct answer.

3 (3 marks)

$$y = \frac{x^6 - 6x^3 + 10}{2x^3}$$

Work out $\frac{dy}{dx}$

Split the fraction: divide **each** term on the top by $2x^3$.

$$\frac{x^6}{2x^3} = \frac{1}{2}x^3$$

$$\frac{-6x^3}{2x^3} = -3$$

$$\frac{10}{2x^3} = 5x^{-3}$$

$$\text{So } y = \frac{1}{2}x^3 - 3 + 5x^{-3}$$

Differentiate each term:

$$\frac{dy}{dx} = \frac{3}{2}x^2 - 15x^{-4}$$

Answer: $\frac{dy}{dx} = \frac{3}{2}x^2 - 15x^{-4}$ (or $1.5x^2 - \frac{15}{x^4}$)

Tip: Never differentiate the top and the bottom separately. Divide every term by $2x^3$ first (subtract the powers when you divide), then differentiate.

Marks: 1 mark for two of the three terms of $\frac{1}{2}x^3 - 3 + 5x^{-3}$. 1 mark for differentiating at least two terms correctly. 1 mark for the fully correct answer.

4 (3 marks)

A curve has equation $y = x(3x - 5) + \frac{16}{x^2}$

Work out the gradient of the curve when $x = -2$

The gradient of a curve is given by $\frac{dy}{dx}$.

Expand and write each term as a power of x : $y = 3x^2 - 5x + 16x^{-2}$

$$\frac{dy}{dx} = 6x - 5 - 32x^{-3}$$

Substitute $x = -2$, using brackets:

$$6(-2) - 5 - \frac{32}{(-2)^3} = -12 - 5 - \frac{32}{-8}$$

$$= -12 - 5 + 4$$

$$= -13$$

Answer: -13

Tip: Substitute into $\frac{dy}{dx}$, not into y . Put -2 in brackets: $(-2)^3 = -8$, so $-\frac{32}{-8} = +4$.

Marks: 1 mark for writing y as $3x^2 - 5x + 16x^{-2}$. 1 mark for $\frac{dy}{dx} = 6x - 5 - 32x^{-3}$. 1 mark for -13 .

5 (3 marks)

A curve has equation $y = 2x^3 - 7x + \frac{3}{x}$

(a) Work out $\frac{d^2y}{dx^2}$

Write $\frac{3}{x}$ as $3x^{-1}$: $y = 2x^3 - 7x + 3x^{-1}$

Differentiate once: $\frac{dy}{dx} = 6x^2 - 7 - 3x^{-2}$

$\frac{d^2y}{dx^2}$ means differentiate $\frac{dy}{dx}$ again:

$$6x^2 \rightarrow 12x, \quad -7 \rightarrow 0, \quad -3x^{-2} \rightarrow (-2) \times (-3)x^{-3} = 6x^{-3}$$

$$\frac{d^2y}{dx^2} = 12x + 6x^{-3}$$

Answer: $\frac{d^2y}{dx^2} = 12x + 6x^{-3}$ (or $12x + \frac{6}{x^3}$)

Tip: Watch the signs with negative powers: $-3x^{-2}$ differentiates to $+6x^{-3}$ (a negative times a negative).

Marks: 1 mark for $\frac{dy}{dx} = 6x^2 - 7 - 3x^{-2}$. 1 mark for $12x + 6x^{-3}$.

(b) Circle the rate of change of the gradient of the curve at the point where $x = -1$

Options: -18 , -4 , 2 , 18

The gradient is $\frac{dy}{dx}$, so the rate of change of the gradient is $\frac{d^2y}{dx^2}$.

$$\text{Substitute } x = -1: 12(-1) + \frac{6}{(-1)^3} = -12 - 6 = -18$$

(-4 is the gradient $\frac{dy}{dx}$ at $x = -1$, and 2 is the value of y .)

Answer: -18

Tip: $\frac{d^2y}{dx^2}$ measures the rate of change of the gradient. Do not stop at $\frac{dy}{dx}$.

6 (3 marks)

A curve has equation $y = kx^2 - \frac{6}{x}$ where k is a constant.

The gradient of the curve at the point where $x = -2$ is 9.5

Work out the value of k .

Write $\frac{6}{x}$ as $6x^{-1}$: $y = kx^2 - 6x^{-1}$

$$\frac{dy}{dx} = 2kx + 6x^{-2}$$

The gradient at $x = -2$ is 9.5, so substitute $x = -2$ into $\frac{dy}{dx}$:

$$2k(-2) + \frac{6}{(-2)^2} = 9.5$$

$$-4k + 1.5 = 9.5$$

$$-4k = 8$$

$$k = -2$$

Answer: $k = -2$

Tip: The gradient comes from $\frac{dy}{dx}$. Substituting $x = -2$ into the equation of the curve and putting $y = 9.5$ scores nothing. Note that $-6x^{-1}$ differentiates to $+6x^{-2}$.

Marks: 1 mark for $\frac{dy}{dx} = 2kx + 6x^{-2}$. 1 mark for substituting $x = -2$ and putting it equal to 9.5. 1 mark for $k = -2$.

7 (4 marks)

A curve has equation $y = 2x^3 + 3x^2 - 18x + 5$

Use calculus to find the x -coordinates of all the points on the curve where the rate of change of y with respect to x is 18

$$\frac{dy}{dx} = 6x^2 + 6x - 18$$

Put the rate of change equal to 18: $6x^2 + 6x - 18 = 18$

Rearrange: $6x^2 + 6x - 36 = 0$

Divide every term by 6: $x^2 + x - 6 = 0$

Factorise: $(x + 3)(x - 2) = 0$

$x = -3$ or $x = 2$

Answer: $x = -3$ and $x = 2$

Tip: "Rate of change of y with respect to x " means $\frac{dy}{dx}$, so differentiate first; do not put $y = 18$. Rearrange to $\dots = 0$ before factorising, and give both values of x .

Marks: 1 mark for $\frac{dy}{dx} = 6x^2 + 6x - 18$. 1 mark for putting it equal to 18 and rearranging to $= 0$. 1 mark for factorising (or the formula). 1 mark for both values.

8 (5 marks)

A curve has equation $y = ax^2 + \frac{b}{x}$ where a and b are constants.

The point $(2, 10)$ lies on the curve.

The gradient of the curve at the point where $x = -1$ is -2

Work out the value of a and the value of b .

The point $(2, 10)$ is on the curve, so substitute $x = 2$ and $y = 10$ into the equation of the curve:

$$a(2)^2 + \frac{b}{2} = 10$$

$$4a + \frac{b}{2} = 10. \text{ Multiply by 2: } 8a + b = 20 \quad (1)$$

Write $y = ax^2 + bx^{-1}$, so $\frac{dy}{dx} = 2ax - bx^{-2}$

$$\text{At } x = -1: 2a(-1) - \frac{b}{(-1)^2} = -2$$

$$-2a - b = -2, \text{ so } 2a + b = 2 \quad (2)$$

Subtract equation (2) from equation (1): $6a = 18$, so $a = 3$

Substitute into (2): $6 + b = 2$, so $b = -4$

Check in (1): $24 + (-4) = 20$ (correct)

Answer: $a = 3, b = -4$

Tip: Use the right equation for each fact: a point **on the curve** goes into $y = \dots$; a **gradient** goes into $\frac{dy}{dx}$. Bracket the negative: $(-1)^2 = +1$, so $-\frac{b}{(-1)^2} = -b$.

Marks: 1 mark for $4a + \frac{b}{2} = 10$. 1 mark for $\frac{dy}{dx} = 2ax - bx^{-2}$. 1 mark for $-2a - b = -2$. 1 mark for solving the two equations. 1 mark for both values.

9 (4 marks)

$$y = \frac{x^5 - 8x^2 + 4}{2x^2}$$

Work out the value of $\frac{d^2y}{dx^2}$ when $x = -2$

Divide each term on the top by $2x^2$:

$$y = \frac{1}{2}x^3 - 4 + 2x^{-2}$$

$$\frac{dy}{dx} = \frac{3}{2}x^2 - 4x^{-3}$$

$$\frac{d^2y}{dx^2} = 3x + 12x^{-4}$$

$$\text{Substitute } x = -2: 3(-2) + \frac{12}{(-2)^4}$$

$$= -6 + \frac{12}{16}$$

$$= -6 + 0.75 = -5.25$$

Answer: -5.25 (or $-\frac{21}{4}$)

Tip: Simplify the fraction into separate terms before you differentiate. $\frac{4}{2x^2} = 2x^{-2}$, not $8x^{-2}$.
Then $(-2)^4 = +16$.

Marks: 1 mark for $y = \frac{1}{2}x^3 - 4 + 2x^{-2}$ (two terms correct is enough). 1 mark for $\frac{dy}{dx}$. 1 mark for $\frac{d^2y}{dx^2} = 3x + 12x^{-4}$. 1 mark for -5.25 .
