

Composite and Inverse Functions

Solutions

For AQA Level 2 Certificate in Further Mathematics (8365)

Each question is shown in grey, followed by the working, one step per line. The final answer is in a box. *Marks* shows how the marks are usually given in the exam, and **Tip** points out a common mistake.

1 (4 marks)

$$f(x) = 5x - 2 \quad \text{and} \quad g(x) = x^2 + 3$$

(a) Work out $fg(-1)$

$fg(-1)$ means do g first, then f .

$$g(-1) = (-1)^2 + 3 = 4$$

Now put 4 into f :

$$f(4) = 5 \times 4 - 2 = 18$$

Answer: 18

Tip: Work from the inside out: the function next to the x (here g) acts first. $(-1)^2 = 1$, not -1 .

Marks: 1 mark for $g(-1) = 4$. 1 mark for 18.

(b) Solve $f(x) = -17$

$f(x) = -17$ means the **output** is -17 .

$$\text{So solve } 5x - 2 = -17$$

$$\text{Add 2: } 5x = -15$$

$$\text{Divide by 5: } x = -3$$

Check: $5 \times (-3) - 2 = -17$ ✓

Answer: $x = -3$

Tip: Don't substitute -17 for x . Set the whole function equal to -17 and solve. Take care with the negatives: $-17 + 2 = -15$.

Marks: 1 mark for $5x - 2 = -17$. 1 mark for $x = -3$.

2 (1 mark)

$$f(x) = x^3 + 1$$

Circle the expression for $f(2x)$

Options: $2x^3 + 1$, $2x^3 + 2$, $8x^3 + 1$, $8x^3 + 2$

Replace every x in $x^3 + 1$ with $(2x)$, in a bracket.

$$f(2x) = (2x)^3 + 1$$

$$(2x)^3 = 2^3 \times x^3 = 8x^3$$

$$\text{So } f(2x) = 8x^3 + 1$$

Answer: $8x^3 + 1$

Tip: $f(2x)$ is not $2 \times f(x)$ (that gives $2x^3 + 2$). Put the $2x$ in a bracket before cubing.

3 (2 marks)

$$f(x) = \frac{7 - 2x}{5}$$

Work out $f^{-1}(x)$

Write $y = \frac{7 - 2x}{5}$ and make x the subject.

$$\text{Multiply by 5: } 5y = 7 - 2x$$

$$\text{Add } 2x \text{ and subtract } 5y: 2x = 7 - 5y$$

$$\text{Divide by 2: } x = \frac{7 - 5y}{2}$$

Swap back to x : $f^{-1}(x) = \frac{7-5x}{2}$

Answer: $f^{-1}(x) = \frac{7-5x}{2}$

Tip: Watch the signs when the x term is negative. Adding $2x$ to both sides first avoids dividing by a negative number. Finish with the answer in terms of x .

Marks: 1 mark for a correct first step, such as $5y = 7 - 2x$. 1 mark for the correct inverse.

4 (3 marks)

$$g(x) = \frac{2x+5}{3x-2} \quad x \neq \frac{2}{3}$$

Show that $g^{-1}(x) = g(x)$

Write $y = \frac{2x+5}{3x-2}$ and make x the subject.

Multiply both sides by $(3x-2)$: $y(3x-2) = 2x+5$

Expand: $3xy - 2y = 2x + 5$

Collect the x terms on one side: $3xy - 2x = 2y + 5$

Factorise: $x(3y-2) = 2y+5$

Divide: $x = \frac{2y+5}{3y-2}$

So $g^{-1}(x) = \frac{2x+5}{3x-2}$, which is the same as $g(x)$.

Answer: $g^{-1}(x) = \frac{2x+5}{3x-2} = g(x)$

Tip: Get every term with x on one side and everything else on the other, then take x out as a factor. End with a sentence saying the inverse is the same as $g(x)$.

Marks: 1 mark for multiplying out, $3xy - 2y = 2x + 5$. 1 mark for collecting and factorising, $x(3y - 2) = 2y + 5$. 1 mark for the inverse and the conclusion.

5 (3 marks)

$$f(x) = (x - 3)^2 + 1 \quad \text{for } x \geq 3$$

(a) Work out $f^{-1}(x)$

Write $y = (x - 3)^2 + 1$ and make x the subject.

Subtract 1: $y - 1 = (x - 3)^2$

Square root: $x - 3 = \sqrt{y - 1}$

Use the positive square root, because $x \geq 3$ means $x - 3 \geq 0$.

Add 3: $x = 3 + \sqrt{y - 1}$

So $f^{-1}(x) = 3 + \sqrt{x - 1}$

Answer: $f^{-1}(x) = 3 + \sqrt{x - 1}$

Tip: The domain $x \geq 3$ makes f one-to-one, which is why you take only the + square root here.

Marks: 1 mark for $x - 3 = \sqrt{y - 1}$ (or $y - 3 = \sqrt{x - 1}$). 1 mark for the correct inverse.

(b) Write down the domain of f^{-1}

The domain of f^{-1} is the range of f .

$$(x - 3)^2 \geq 0, \text{ so } (x - 3)^2 + 1 \geq 1$$

The range of f is $f(x) \geq 1$, so the domain of f^{-1} is $x \geq 1$.

This also matches $\sqrt{x - 1}$, which needs $x - 1 \geq 0$.

Answer: $x \geq 1$

Tip: The inverse undoes f , so its inputs are f 's outputs. The domain and range swap over.

6 (3 marks)

$$f(x) = 3x + 2 \quad \text{and} \quad g(x) = x^2 - 5x$$

Solve $fg(x) = 44$

$$fg(x) = f(x^2 - 5x) = 3(x^2 - 5x) + 2 = 3x^2 - 15x + 2$$

Solve $3x^2 - 15x + 2 = 44$

Subtract 44: $3x^2 - 15x - 42 = 0$

Divide by 3: $x^2 - 5x - 14 = 0$

Factorise: $(x - 7)(x + 2) = 0$

$x = 7$ or $x = -2$

Answer: $x = 7$ or $x = -2$

Tip: $fg(x)$ means put $g(x)$ into f , so multiply all of $x^2 - 5x$ by 3. A quadratic usually has two solutions: give both.

Marks: 1 mark for $3(x^2 - 5x) + 2$. 1 mark for a correct quadratic equal to 0. 1 mark for both solutions.

7 (2 marks)

$f(x) = x + 4$ and $gf(x) = x^2 + 8x + 19$

Find $g(x)$

$gf(x)$ means g acts on $f(x) = x + 4$.

So write $x^2 + 8x + 19$ in terms of $(x + 4)$.

$$(x + 4)^2 = x^2 + 8x + 16$$

$$\text{So } x^2 + 8x + 19 = (x + 4)^2 + 3$$

g squares its input and then adds 3:

$$g(x) = x^2 + 3$$

Answer: $g(x) = x^2 + 3$

Tip: Check your answer: put $x + 4$ into your $g(x)$ and expand. You should get back to $x^2 + 8x + 19$.

Marks: 1 mark for $(x + 4)^2 + 3$. 1 mark for $g(x) = x^2 + 3$.

8 (4 marks)

$$f(x) = 3x - 1 \quad \text{and} \quad g(x) = x^2$$

$$\text{Solve } gf(x) = f^{-1}(x)$$

Give your answers to 2 decimal places.

$$gf(x) = g(3x - 1) = (3x - 1)^2$$

$$\text{Inverse of } f: y = 3x - 1, \text{ so } x = \frac{y+1}{3} \text{ and } f^{-1}(x) = \frac{x+1}{3}$$

$$\text{Solve } (3x - 1)^2 = \frac{x+1}{3}$$

$$\text{Multiply by 3: } 3(3x - 1)^2 = x + 1$$

$$\text{Expand: } 3(9x^2 - 6x + 1) = x + 1, \text{ so } 27x^2 - 18x + 3 = x + 1$$

$$27x^2 - 19x + 2 = 0$$

$$\text{Formula: } x = \frac{19 \pm \sqrt{(-19)^2 - 4 \times 27 \times 2}}{2 \times 27} = \frac{19 \pm \sqrt{145}}{54}$$

$$x = 0.5748 \dots \text{ or } x = 0.1288 \dots$$

$$\text{To 2 decimal places: } x = 0.57 \text{ or } x = 0.13$$

Answer: $x = 0.57$ or $x = 0.13$

Tip: Expand $(3x - 1)^2$ fully: it is $9x^2 - 6x + 1$, not $9x^2 + 1$. Multiply **every** term by 3 when you clear the fraction, and don't round until the very end.

Marks: 1 mark for $f^{-1}(x) = \frac{x+1}{3}$. 1 mark for $(3x - 1)^2$. 1 mark for $27x^2 - 19x + 2 = 0$. 1 mark for both answers.

9 (5 marks)

$$f(x) = ax + b, \text{ where } a \text{ and } b \text{ are constants and } a > 0$$

$$ff(x) = 4x + 9$$

(a) Work out the values of a and b .

$$ff(x) = f(ax + b) = a(ax + b) + b$$

$$= a^2x + ab + b$$

Compare with $4x + 9$.

x terms: $a^2 = 4$, so $a = 2$ (because $a > 0$)

Numbers: $ab + b = 9$, so $2b + b = 9$, $3b = 9$, $b = 3$

Answer: $a = 2$, $b = 3$

Tip: $a^2 = 4$ also gives $a = -2$, but the question says $a > 0$. In $a(ax + b) + b$ don't forget the final $+b$.

Marks: 1 mark for $a(ax + b) + b$. 1 mark for $a = 2$. 1 mark for $b = 3$.

(b) Solve $f(x) = f^{-1}(x)$

$f(x) = 2x + 3$. Inverse: $y = 2x + 3$, so $x = \frac{y-3}{2}$, $f^{-1}(x) = \frac{x-3}{2}$

Solve $2x + 3 = \frac{x-3}{2}$

Multiply by 2: $4x + 6 = x - 3$

$3x = -9$

$x = -3$

Check: $f(-3) = -3$ and $f^{-1}(-3) = \frac{-6}{2} = -3$ ✓

Answer: $x = -3$

Marks: 1 mark for $f^{-1}(x) = \frac{x-3}{2}$. 1 mark for $x = -3$.