

Completing the Square

Solutions

For AQA Level 2 Certificate in Further Mathematics (8365)

Each question is shown in grey, followed by the working, one step per line. The final answer is in a box. *Marks* shows how the marks are usually given in the exam, and **Tip** points out a common mistake.

1 (1 mark)

Circle the coordinates of the minimum point of the curve $y = (x + 4)^2 - 7$

Options: $(4, -7)$, $(-4, -7)$, $(-7, -4)$, $(4, 7)$

$(x + 4)^2$ is never negative, and it equals 0 when $x = -4$.

So the smallest value of y is $0 - 7 = -7$, when $x = -4$.

The minimum point is $(-4, -7)$.

Answer: $(-4, -7)$

Tip: The x -coordinate has the opposite sign to the number in the bracket: $(x + 4)^2$ gives $x = -4$. Write the x -coordinate first.

2 (2 marks)

Write $x^2 + 7x - 3$ in the form $(x + a)^2 + b$

Halve the coefficient of x : $\frac{7}{2}$

$$\left(x + \frac{7}{2}\right)^2 = x^2 + 7x + \frac{49}{4}$$

$$\text{So } x^2 + 7x = \left(x + \frac{7}{2}\right)^2 - \frac{49}{4}$$

$$x^2 + 7x - 3 = \left(x + \frac{7}{2}\right)^2 - \frac{49}{4} - 3$$

$$= \left(x + \frac{7}{2}\right)^2 - \frac{61}{4}$$

Answer: $\left(x + \frac{7}{2}\right)^2 - \frac{61}{4}$ or $(x + 3.5)^2 - 15.25$

Tip: Subtract the square of half the coefficient, $\left(\frac{7}{2}\right)^2 = \frac{49}{4}$, then take off the 3 as well.

Marks: 1 mark for $\left(x + \frac{7}{2}\right)^2$. 1 mark for $-\frac{61}{4}$.

3 (3 marks)

Write $3x^2 - 42x + 143$ in the form $a(x + b)^2 + c$, where a , b and c are integers.

Take out the factor 3 from the x^2 and x terms: $3(x^2 - 14x) + 143$

Complete the square inside the bracket: $x^2 - 14x = (x - 7)^2 - 49$

So we get $3[(x - 7)^2 - 49] + 143$

Multiply out the square bracket: $3(x - 7)^2 - 147 + 143$

$$= 3(x - 7)^2 - 4$$

Answer: $3(x - 7)^2 - 4$

Tip: Take out the 3 first, and keep the sign: it is $3(x - 7)^2$, not $3(x + 7)^2$. The -49 inside the bracket is multiplied by 3 too. Expand your answer to check it.

Marks: 1 mark for $3(x^2 - 14x)$. 1 mark for $3(x - 7)^2$. 1 mark for -4 .

4 (4 marks)

(a) Write $7 + 12x - 2x^2$ in the form $a + b(x + c)^2$, where a , b and c are integers.

Take out -2 from the x^2 and x terms:

$$-2x^2 + 12x = -2(x^2 - 6x)$$

Complete the square inside the bracket: $x^2 - 6x = (x - 3)^2 - 9$

$$\text{So } 7 + 12x - 2x^2 = 7 - 2[(x - 3)^2 - 9]$$

$$= 7 - 2(x - 3)^2 + 18$$

$$= 25 - 2(x - 3)^2$$

Answer: $25 - 2(x - 3)^2$ ($a = 25, b = -2, c = -3$)

Tip: Taking out -2 changes the sign of the x term: $12x = -2 \times (-6x)$. Then $-2 \times (-9) = +18$.

Marks: 1 mark for $-2(x^2 - 6x)$. 1 mark for $-2(x - 3)^2$. 1 mark for 25.

(b) Hence write down the coordinates of the maximum point of the curve

$$y = 7 + 12x - 2x^2$$

$$y = 25 - 2(x - 3)^2$$

$2(x - 3)^2$ is never negative, so y is never more than 25.

$y = 25$ when $(x - 3)^2 = 0$, so when $x = 3$.

Answer: (3, 25)

Tip: In the exam, only about half of students got this type of question right. $(x - 3)^2$ gives $x = +3$, not -3 , and the x -coordinate goes first.

5 (3 marks)

By completing the square, solve $x^2 - 10x + 7 = 0$

Give your answers in the form $p \pm q\sqrt{2}$, where p and q are integers.

Complete the square: $x^2 - 10x = (x - 5)^2 - 25$

$$\text{So } (x - 5)^2 - 25 + 7 = 0$$

$$(x - 5)^2 - 18 = 0$$

$$(x - 5)^2 = 18$$

Square root both sides: $x - 5 = \pm\sqrt{18}$

$$\sqrt{18} = \sqrt{9 \times 2} = 3\sqrt{2}$$

$$x = 5 \pm 3\sqrt{2}$$

Answer: $x = 5 + 3\sqrt{2}$ and $x = 5 - 3\sqrt{2}$

Tip: Once you have $(x - 5)^2 = 18$, don't expand it again: square root both sides and remember $\pm$, which gives two answers.

Marks: 1 mark for $(x - 5)^2 = 18$. 1 mark for $x - 5 = \pm\sqrt{18}$. 1 mark for $5 \pm 3\sqrt{2}$.

6 (3 marks)

The curve $y = 4x^2 + px + q$ has its minimum point at $\left(\frac{3}{4}, 5\right)$

Work out the value of p and the value of q .

The coefficient of x^2 is 4, so in completed-square form the curve is $y = 4(x + b)^2 + c$

The minimum point is $\left(\frac{3}{4}, 5\right)$, so $b = -\frac{3}{4}$ and $c = 5$:

$$y = 4\left(x - \frac{3}{4}\right)^2 + 5$$

$$\text{Expand: } 4\left(x^2 - \frac{3}{2}x + \frac{9}{16}\right) + 5$$

$$= 4x^2 - 6x + \frac{9}{4} + 5$$

$$= 4x^2 - 6x + \frac{29}{4}$$

Answer: $p = -6$, $q = \frac{29}{4}$

Tip: Keep the 4 outside the bracket: $(x - \frac{3}{4})^2 + 5$ on its own would give x^2 , not $4x^2$. Remember to multiply the $\frac{9}{16}$ by 4.

Marks: 1 mark for $4\left(x - \frac{3}{4}\right)^2 + 5$. 1 mark for $p = -6$. 1 mark for $q = \frac{29}{4}$.

7 (4 marks)

$$y = 2x^3 + 9x^2 + 15x - 4$$

- (a) Work out $\frac{dy}{dx}$ and write it in the form $6(x + a)^2 + b$

Differentiate: $\frac{dy}{dx} = 6x^2 + 18x + 15$

Take out the factor 6 from the x^2 and x terms: $6(x^2 + 3x) + 15$

Complete the square: $x^2 + 3x = \left(x + \frac{3}{2}\right)^2 - \frac{9}{4}$

$$\text{So } \frac{dy}{dx} = 6 \left[\left(x + \frac{3}{2}\right)^2 - \frac{9}{4} \right] + 15$$

$$= 6 \left(x + \frac{3}{2}\right)^2 - \frac{27}{2} + 15$$

$$= 6 \left(x + \frac{3}{2}\right)^2 + \frac{3}{2}$$

Answer: $\frac{dy}{dx} = 6 \left(x + \frac{3}{2}\right)^2 + \frac{3}{2}$

Tip: Differentiate first: the -4 disappears and $9x^2$ becomes $18x$. When you multiply out, $6 \times \left(-\frac{9}{4}\right) = -\frac{27}{2}$.

Marks: 1 mark for $6x^2 + 18x + 15$. 1 mark for $6 \left(x + \frac{3}{2}\right)^2$. 1 mark for $+\frac{3}{2}$.

- (b) Hence explain why y is an increasing function for all values of x .

$$\left(x + \frac{3}{2}\right)^2 \geq 0 \text{ for all } x, \text{ so } 6 \left(x + \frac{3}{2}\right)^2 \geq 0$$

$$\text{So } \frac{dy}{dx} \geq 0 + \frac{3}{2} > 0 \text{ for all values of } x.$$

The gradient is always positive, so y is an increasing function.

Answer: $\frac{dy}{dx} = 6\left(x + \frac{3}{2}\right)^2 + \frac{3}{2} \geq \frac{3}{2} > 0$ for all x , so y is increasing

Tip: Say **why** $\frac{dy}{dx} > 0$: a square is never negative, so $\frac{dy}{dx}$ is at least $\frac{3}{2}$. Testing a few values of x does not show it for **all** x .

8 (4 marks)

$y = x^2 - 2kx + k + 6$, where k is a constant.

(a) By completing the square, show that the minimum value of y is $6 + k - k^2$

Halve the coefficient of x : $x^2 - 2kx = (x - k)^2 - k^2$

So $y = (x - k)^2 - k^2 + k + 6$

$(x - k)^2 \geq 0$, and it equals 0 when $x = k$.

So the minimum value of y is $-k^2 + k + 6 = 6 + k - k^2$

Answer: $y = (x - k)^2 + 6 + k - k^2$, so the minimum value is $6 + k - k^2$

Tip: Treat k just like a number: half of $-2k$ is $-k$, and you subtract $(-k)^2 = k^2$.

Marks: 1 mark for $(x - k)^2 - k^2$. 1 mark for completing the argument.

(b) The curve touches the x -axis.

Work out the two possible values of k .

The curve touches the x -axis, so its minimum point is on the x -axis.

So the minimum value of y is 0:

$$6 + k - k^2 = 0$$

$$k^2 - k - 6 = 0$$

$$(k - 3)(k + 2) = 0$$

$$k = 3 \text{ or } k = -2$$

Answer: $k = 3$ or $k = -2$

Tip: “Touches” means the curve meets the axis at its turning point only, so the minimum value is exactly 0.

Marks: 1 mark for $6 + k - k^2 = 0$. 1 mark for both values.
