

Binomial Expansion

Solutions

For AQA Level 2 Certificate in Further Mathematics (8365)

Each question is shown in grey, followed by the working, one step per line. The final answer is in a box. *Marks* shows how the marks are usually given in the exam, and **Tip** points out a common mistake.

1 (1 mark)

Here are the first five rows of Pascal's triangle.

$$\begin{array}{ccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

Write down the row of Pascal's triangle used to expand $(a + b)^6$

Each number is the sum of the two numbers above it.

Next row (for power 5): 1 5 10 10 5 1

Row after that (for power 6): 1 6 15 20 15 6 1

Answer: 1 6 15 20 15 6 1

Tip: The row for power n starts 1, n , ... and has $n + 1$ numbers. The row 1 4 6 4 1 is for power 4, so you need two more rows.

2 (3 marks)

Expand and simplify fully $(x - 3)^4$

Power 4 row of Pascal's triangle: 1 4 6 4 1

Powers of x go down from 4; powers of (-3) go up from 0.

$$1 \times x^4 = x^4$$

$$4 \times x^3 \times (-3) = -12x^3$$

$$6 \times x^2 \times (-3)^2 = 6 \times 9 \times x^2 = 54x^2$$

$$4 \times x \times (-3)^3 = 4 \times (-27) \times x = -108x$$

$$1 \times (-3)^4 = 81$$

Answer: $x^4 - 12x^3 + 54x^2 - 108x + 81$

Tip: Keep the minus sign with the 3: use $(-3)^2 = 9$, $(-3)^3 = -27$, $(-3)^4 = 81$. The signs then alternate +, −, +, −, +.

Marks: 1 mark for using 1, 4, 6, 4, 1. 1 mark for at least three correct terms. 1 mark for the fully correct expansion.

3 (4 marks)

Work out the coefficient of x^3 in the expansion of

(a) $(6 - x)^5$

Power 5 row: 1 5 10 10 5 1

The x^3 term uses $(-x)^3$, so the 6 is raised to the power $5 - 3 = 2$.

Coefficient = 10 (the fourth number in the row)

$$\begin{aligned} \text{Term} &= 10 \times 6^2 \times (-x)^3 \\ &= 10 \times 36 \times (-x^3) = -360x^3 \end{aligned}$$

Answer: -360

Tip: $(-x)^3 = -x^3$, so the coefficient is negative. The coefficient is the number only, so write -360 , not $-360x^3$.

Marks: 1 mark for $10 \times 6^2 \times (-x)^3$ or equivalent. 1 mark for -360 .

(b) $(2x - 3)^5$

The x^3 term needs $(2x)^3$, so (-3) is raised to the power $5 - 3 = 2$.

Coefficient from the row: 10

$$\text{Term} = 10 \times (2x)^3 \times (-3)^2$$

$$= 10 \times 8x^3 \times 9$$

$$= 720x^3$$

Answer: 720

Tip: Put a bracket round the $2x$: $(2x)^3 = 8x^3$, not $2x^3$. Multiply the parts of the term together; do not add them.

Marks: 1 mark for $10 \times (2x)^3 \times (-3)^2$ or equivalent. 1 mark for 720.

4 (3 marks)

Expand and simplify $(2 + x)^5 - (2 - x)^5$

Power 5 row: 1 5 10 10 5 1

$$(2 + x)^5 = 32 + 80x + 80x^2 + 40x^3 + 10x^4 + x^5$$

$$(2 - x)^5 = 32 - 80x + 80x^2 - 40x^3 + 10x^4 - x^5$$

Subtract: the terms with even powers of x cancel, and the terms with odd powers double.

$$80x - (-80x) = 160x$$

$$40x^3 - (-40x^3) = 80x^3$$

$$x^5 - (-x^5) = 2x^5$$

Answer: $160x + 80x^3 + 2x^5$

Tip: In $(2 - x)^5$ the odd powers of x are negative. Subtracting a negative term adds it, so those terms double rather than cancel.

Marks: 1 mark for a correct expansion of $(2 + x)^5$ or $(2 - x)^5$. 1 mark for both correct (or for spotting that only the odd powers are left). 1 mark for the correct answer.

5 (4 marks)

Expand and simplify fully $\left(4 + \frac{x}{2}\right)^4$

Power 4 row: 1 4 6 4 1

$$1 \times 4^4 = 256$$

$$4 \times 4^3 \times \frac{x}{2} = 4 \times 64 \times \frac{x}{2} = 128x$$

$$6 \times 4^2 \times \left(\frac{x}{2}\right)^2 = 6 \times 16 \times \frac{x^2}{4} = 24x^2$$

$$4 \times 4 \times \left(\frac{x}{2}\right)^3 = 16 \times \frac{x^3}{8} = 2x^3$$

$$1 \times \left(\frac{x}{2}\right)^4 = \frac{x^4}{16}$$

Answer: $256 + 128x + 24x^2 + 2x^3 + \frac{x^4}{16}$

Tip: Square the whole fraction: $\left(\frac{x}{2}\right)^2 = \frac{x^2}{4}$, not $\frac{x^2}{2}$. The last term can also be written as $\frac{1}{16}x^4$ or $0.0625x^4$.

Marks: 1 mark for using 1, 4, 6, 4, 1 with the correct powers. 1 mark for at least two correct terms. 1 mark for at least four correct terms. 1 mark for the fully correct expansion.

6 (3 marks)

In the expansion of $(3 - kx)^5$, the coefficient of x^4 is 1215

Work out both possible values of k .

Power 5 row: 1 5 10 10 5 1

The x^4 term is $5 \times 3 \times (-kx)^4$

$(-kx)^4 = k^4x^4$ (an even power, so it is positive)

So the coefficient is $15k^4$

$$15k^4 = 1215$$

$$k^4 = 81$$

$$k = 3 \text{ or } k = -3$$

Answer: $k = 3$ and $k = -3$

Tip: Raise the whole term to the power: $(-kx)^4 = k^4x^4$, not kx^4 . An even power means k can be positive or negative, so give both values.

Marks: 1 mark for $5 \times 3 \times (-kx)^4$ or $15k^4$. 1 mark for $15k^4 = 1215$. 1 mark for both values.

7 (4 marks)

a is a non-zero constant.

The coefficient of x^3 in the expansion of $(2 + ax)^5$ is equal to the coefficient of x^2 in the expansion of $(a - 4x)^4$

Work out the value of a .

In $(2 + ax)^5$: the x^3 term is $10 \times 2^2 \times (ax)^3 = 40a^3x^3$

So the coefficient of x^3 is $40a^3$

In $(a - 4x)^4$: the x^2 term is $6 \times a^2 \times (-4x)^2 = 6 \times a^2 \times 16x^2 = 96a^2x^2$

So the coefficient of x^2 is $96a^2$

Equate the coefficients: $40a^3 = 96a^2$

a is not zero, so divide both sides by a^2 : $40a = 96$

$$a = \frac{96}{40} = 2.4$$

Answer: $a = 2.4$ ($= \frac{12}{5}$)

Tip: Use the coefficients only, without the x terms. Bracket the whole term: $(ax)^3 = a^3x^3$ and $(-4x)^2 = 16x^2$. You can divide by a^2 because you are told $a \neq 0$.

Marks: 1 mark for $40a^3$. 1 mark for $96a^2$. 1 mark for forming $40a^3 = 96a^2$. 1 mark for $a = 2.4$.

8 (4 marks)

In the expansion of $(1 + ax)^6$

the coefficient of x^2 is 48 more than the coefficient of x

Work out the two possible values of a .

You must show your working.

Power 6 row: 1 6 15 20 15 6 1

The x term is $6 \times (ax) = 6ax$, so the coefficient of x is $6a$

The x^2 term is $15 \times (ax)^2 = 15a^2x^2$, so the coefficient of x^2 is $15a^2$

$$15a^2 = 6a + 48$$

$$15a^2 - 6a - 48 = 0$$

$$\text{Divide by 3: } 5a^2 - 2a - 16 = 0$$

$$\text{Factorise: } (5a + 8)(a - 2) = 0$$

$$a = 2 \text{ or } a = -\frac{8}{5}$$

Answer: $a = 2$ and $a = -1.6$

Tip: “48 more than” means coefficient of $x^2 =$ coefficient of $x + 48$. Check both answers: when $a = 2$, $15 \times 4 = 60$ and $6 \times 2 + 48 = 60$.

Marks: 1 mark for $6a$ and $15a^2$. 1 mark for $15a^2 = 6a + 48$. 1 mark for a correct method to solve the quadratic. 1 mark for both values.
