

Algebraic Proof

Solutions

For AQA Level 2 Certificate in Further Mathematics (8365)

Each question is shown in grey, followed by the working, one step per line. The final answer is in a box. *Marks* shows how the marks are usually given in the exam, and **Tip** points out a common mistake.

1 (1 mark)

n is an integer.

Circle the expression that is a multiple of 4 for every value of n .

Options: $4n + 2$, $2n + 4$, $2(2n + 6)$, $n^2 + 4$

$4n + 2 = 2(2n + 1)$: this is 2 more than a multiple of 4, so it is never a multiple of 4.

$2n + 4$: when $n = 1$ it is 6, which is not a multiple of 4.

$2(2n + 6) = 4n + 12 = 4(n + 3)$: 4 times an integer, so always a multiple of 4.

$n^2 + 4$: when $n = 1$ it is 5, which is not a multiple of 4.

Answer: $2(2n + 6)$

Tip: To show an expression is always a multiple of 4, write it as $4 \times$ (an integer). One counter-example is enough to rule an expression out.

2 (2 marks)

n is an integer.

Show that $4(2n + 3) - 2(n - 3)$ is a multiple of 6

Expand both brackets: $4(2n + 3) = 8n + 12$ and $2(n - 3) = 2n - 6$

Subtract, keeping the second expansion in a bracket: $8n + 12 - (2n - 6)$

$$= 8n + 12 - 2n + 6$$

$$= 6n + 18$$

$$= 6(n + 3)$$

$n + 3$ is an integer, so $6(n + 3)$ is a multiple of 6.

Answer: $6(n + 3)$, which is a multiple of 6

Tip: $-2 \times (-3) = +6$. Subtracting $2n - 6$ changes **both** signs, so the -6 becomes $+6$.

Marks: 1 mark for $6n + 18$. 1 mark for $6(n + 3)$ with a statement that it is a multiple of 6.

3 (3 marks)

The n th term of a sequence is $6n + 3$

Prove that the sum of any two consecutive terms of the sequence is a multiple of 12

Take any term, the n th term: $6n + 3$

The next term is the $(n + 1)$ th term: $6(n + 1) + 3 = 6n + 9$

Sum: $(6n + 3) + (6n + 9) = 12n + 12$

$= 12(n + 1)$

$n + 1$ is an integer, so the sum is a multiple of 12.

Answer: $12(n + 1)$, which is a multiple of 12

Tip: The next term is found by replacing n with $n + 1$, not by adding 1 to the term ($6n + 4$ is wrong). Adding the common difference, 6, also works.

Marks: 1 mark for the next term $6n + 9$ (or $6(n + 1) + 3$). 1 mark for $12n + 12$. 1 mark for $12(n + 1)$ with a conclusion.

4 (3 marks)

a is an odd number and b is an even number.

Prove that $a^2 + ab + b^2$ is odd.

a and b can be any odd and even numbers, so use two different letters:

Let $a = 2m + 1$ and $b = 2n$, where m and n are integers.

$$a^2 = (2m + 1)^2 = 4m^2 + 4m + 1$$

$$ab = (2m + 1)(2n) = 4mn + 2n$$

$$b^2 = (2n)^2 = 4n^2$$

$$a^2 + ab + b^2 = 4m^2 + 4m + 4mn + 2n + 4n^2 + 1$$

$$= 2(2m^2 + 2m + 2mn + n + 2n^2) + 1$$

$2(2m^2 + 2m + 2mn + n + 2n^2)$ is even, so adding 1 gives an odd number.

Answer: $2(2m^2 + 2m + 2mn + n + 2n^2) + 1$, which is odd

Tip: Use different letters for a and b : writing $a = 2n + 1$ and $b = 2n$ only proves it for numbers next to each other. Bracket $(2n)$ before squaring: $(2n)^2 = 4n^2$.

Marks: 1 mark for using $2m + 1$ and $2n$ (different letters). 1 mark for a correct expansion. 1 mark for writing it as $2(\dots) + 1$ with a conclusion.

5 (3 marks)

Prove that $2x^2 - 20x + 53 > 0$ for all values of x .

Complete the square. Take out the factor 2 from the x terms:

$$2x^2 - 20x + 53 = 2(x^2 - 10x) + 53$$

$$x^2 - 10x = (x - 5)^2 - 25$$

$$\text{So } 2x^2 - 20x + 53 = 2((x - 5)^2 - 25) + 53$$

$$= 2(x - 5)^2 - 50 + 53$$

$$= 2(x - 5)^2 + 3$$

A square is never negative, so $(x - 5)^2 \geq 0$ and $2(x - 5)^2 \geq 0$.

So $2(x - 5)^2 + 3 \geq 3$, which is greater than 0 for all values of x .

Answer: $2(x - 5)^2 + 3 \geq 3 > 0$

Tip: Multiply the -25 by the 2 outside the bracket: $2 \times (-25) = -50$. The final mark needs the reason: a squared term can never be negative.

Marks: 1 mark for $2(x - 5)^2$. 1 mark for $2(x - 5)^2 + 3$. 1 mark for explaining that $(x - 5)^2 \geq 0$, so the expression is always at least 3.

6 (3 marks)

k is an integer.

Prove that $(4k + 3)(k + 6) - (2k + 5)^2$ is a multiple of 7

Expand the first product: $(4k + 3)(k + 6) = 4k^2 + 24k + 3k + 18 = 4k^2 + 27k + 18$

Expand the square: $(2k + 5)^2 = 4k^2 + 20k + 25$

Subtract, keeping the second expansion in a bracket:

$$4k^2 + 27k + 18 - (4k^2 + 20k + 25)$$

$$= 4k^2 + 27k + 18 - 4k^2 - 20k - 25$$

$$= 7k - 7$$

$$= 7(k - 1)$$

$k - 1$ is an integer, so this is a multiple of 7.

Answer: $7(k - 1)$, which is a multiple of 7

Tip: Put the whole of $(2k + 5)^2$ in a bracket before you subtract, so that **every** sign changes. A common wrong answer is $7k + 43$, from adding the 25.

Marks: 1 mark for expanding both products correctly. 1 mark for $7k - 7$. 1 mark for $7(k - 1)$ with a conclusion.

7 (3 marks)

Prove that the difference between the cubes of two consecutive integers is always odd.

Let the two consecutive integers be n and $n + 1$.

$$(n + 1)^2 = n^2 + 2n + 1, \text{ so } (n + 1)^3 = (n + 1)(n^2 + 2n + 1) = n^3 + 3n^2 + 3n + 1$$

$$\text{Difference: } (n + 1)^3 - n^3 = n^3 + 3n^2 + 3n + 1 - n^3 = 3n^2 + 3n + 1$$

$$\text{Factorise the first two terms: } 3n^2 + 3n + 1 = 3n(n + 1) + 1$$

n and $n + 1$ are consecutive integers, so one of them is even, which makes $n(n + 1)$ even.

So $3n(n + 1)$ is even, and $3n(n + 1) + 1$ is odd.

Answer: $3n(n + 1) + 1$, which is odd

Tip: $3n^2 + 3n + 1$ alone is not enough. Explain why $3n^2 + 3n$ is even, e.g. $3n(n + 1)$ contains two consecutive integers, one of which is even.

Marks: 1 mark for expanding $(n + 1)^3$ correctly. 1 mark for $3n^2 + 3n + 1$. 1 mark for a full reason why $3n(n + 1) + 1$ is odd.

8 (4 marks)

(a) Prove that $n(n + 1)(n + 2)(n + 3) + 1 = (n^2 + 3n + 1)^2$

Work on each side separately.

Left side. Pair the outer brackets and the inner brackets:

$$n(n + 3) = n^2 + 3n \quad \text{and} \quad (n + 1)(n + 2) = n^2 + 3n + 2$$

$$(n^2 + 3n)(n^2 + 3n + 2) = n^4 + 3n^3 + 2n^2 + 3n^3 + 9n^2 + 6n$$

$$= n^4 + 6n^3 + 11n^2 + 6n$$

So the left side is $n^4 + 6n^3 + 11n^2 + 6n + 1$

$$\text{Right side: } (n^2 + 3n + 1)(n^2 + 3n + 1) = n^4 + 3n^3 + n^2 + 3n^3 + 9n^2 + 3n + n^2 + 3n + 1$$

$$= n^4 + 6n^3 + 11n^2 + 6n + 1$$

Both sides are equal, so the identity is true.

Answer: Both sides expand to $n^4 + 6n^3 + 11n^2 + 6n + 1$

Tip: Expand each side on its own and show that both give the same expression. Don't write the two sides as an equation and do the same thing to both.

Marks: 1 mark for multiplying two pairs of brackets correctly. 1 mark for the left side $n^4 + 6n^3 + 11n^2 + 6n + 1$. 1 mark for expanding the right side to the same expression, with a conclusion.

- (b) Hence prove that one more than the product of any four consecutive integers is a square number.

Any four consecutive integers can be written as $n, n + 1, n + 2, n + 3$, where n is an integer.

By part (a), one more than their product is $(n^2 + 3n + 1)^2$.

$n^2 + 3n + 1$ is an integer, so this is a square number.

Answer: It equals $(n^2 + 3n + 1)^2$, the square of an integer

Tip: Say why it is a square number: $n^2 + 3n + 1$ is an integer. Testing examples such as $1 \times 2 \times 3 \times 4 + 1 = 25$ is not a proof.
