

Turning Points by Completing the Square

Solutions

For Pearson Edexcel GCSE Mathematics (1MA1) • Higher tier

Each question is shown in grey, followed by the working, one step per line. The final answer is in a box. *Marks* shows how the marks are usually given in the exam, and **Tip** points out a common mistake.

1 (2 marks)

Write down the coordinates of the turning point on the graph of

(a) $y = (x - 4)^2 + 9$

$(x - 4)^2$ is never negative. It is smallest (equal to 0) when $x = 4$.

Then $y = 0 + 9 = 9$

Answer: (4, 9)

(b) $y = (x + 10)^2 - 3$

$(x + 10)^2$ is smallest (equal to 0) when $x + 10 = 0$, so when $x = -10$.

Then $y = 0 - 3 = -3$

Answer: (-10, -3)

Tip: The x -coordinate is the value that makes the bracket zero. For $(x + 10)^2$ it is -10 , not 10.

2 (3 marks)

(a) Write $x^2 + 4x - 1$ in the form $(x + a)^2 + b$

Halve the coefficient of x : $4 \div 2 = 2$

$(x + 2)^2 = x^2 + 4x + 4$

$$x^2 + 4x - 1 = (x + 2)^2 - 4 - 1$$

$$= (x + 2)^2 - 5$$

Answer: $(x + 2)^2 - 5$

Marks: 1 mark for $(x + 2)^2$. 1 mark for the correct answer.

- (b) Write down the coordinates of the turning point of the graph of $y = x^2 + 4x - 1$

$$y = (x + 2)^2 - 5$$

$(x + 2)^2$ is smallest (equal to 0) when $x = -2$.

Then $y = -5$

Answer: $(-2, -5)$

3 (3 marks)

The graph of $y = x^2 - 5x + 2$ has a turning point.
Find the coordinates of the turning point.

Complete the square: halve -5 to get $-\frac{5}{2}$.

$$\left(x - \frac{5}{2}\right)^2 = x^2 - 5x + \frac{25}{4}$$

$$x^2 - 5x + 2 = \left(x - \frac{5}{2}\right)^2 - \frac{25}{4} + 2$$

$$-\frac{25}{4} + \frac{8}{4} = -\frac{17}{4}$$

$$\text{So } y = \left(x - \frac{5}{2}\right)^2 - \frac{17}{4}$$

The bracket is zero when $x = \frac{5}{2}$, and then $y = -\frac{17}{4}$.

Answer: $\left(\frac{5}{2}, -\frac{17}{4}\right)$

Tip: Halving an odd number gives a fraction. Keep it as $\frac{5}{2}$ (or 2.5). Do not round it.

Marks: 1 mark for $\left(x - \frac{5}{2}\right)^2$. 1 mark for $\left(x - \frac{5}{2}\right)^2 - \frac{17}{4}$. 1 mark for the correct answer.

4 (3 marks)

Find the coordinates of the turning point of the curve $y = 3x^2 + 6x + 8$

Take out the factor 3 from the x^2 and x terms: $3(x^2 + 2x) + 8$

Complete the square: $x^2 + 2x = (x + 1)^2 - 1$

So $y = 3[(x + 1)^2 - 1] + 8$

$$= 3(x + 1)^2 - 3 + 8$$

$$= 3(x + 1)^2 + 5$$

The bracket is zero when $x = -1$, and then $y = 5$.

Answer: $(-1, 5)$

Marks: 1 mark for $3(x^2 + 2x)$. 1 mark for $3(x + 1)^2 + 5$. 1 mark for the correct answer.

5 (3 marks)

The curve $y = 8 + 6x - x^2$ has a maximum point M .

Find the coordinates of M .

Take out -1 from the x^2 and x terms: $y = -(x^2 - 6x) + 8$

Complete the square: $x^2 - 6x = (x - 3)^2 - 9$

So $y = -[(x - 3)^2 - 9] + 8$

$$= -(x - 3)^2 + 9 + 8$$

$$= 17 - (x - 3)^2$$

$(x - 3)^2$ is never negative, so y is never more than 17.

The maximum is $y = 17$, when $x = 3$.

Answer: $(3, 17)$

Marks: 1 mark for $-(x^2 - 6x)$. 1 mark for $17 - (x - 3)^2$. 1 mark for the correct answer.

6 (3 marks)

A curve has equation $y = x^2 + px + q$

The turning point of the curve is $(-3, 7)$

Find the value of p and the value of q .

A turning point at $(-3, 7)$ means the completed square form is $y = (x + 3)^2 + 7$.

Expand: $(x + 3)^2 = x^2 + 6x + 9$

So $y = x^2 + 6x + 9 + 7 = x^2 + 6x + 16$

Compare with $x^2 + px + q$:

$p = 6$ and $q = 16$

Answer: $p = 6$, $q = 16$

Tip: An x -coordinate of -3 gives the bracket $(x + 3)$, because $x = -3$ makes $x + 3 = 0$.

Marks: 1 mark for $(x + 3)^2 + 7$. 1 mark for $p = 6$. 1 mark for $q = 16$.

7 (4 marks)

A ball is thrown upwards from the top of a wall.

Its height, h metres, above the ground t seconds after it is thrown is given by

$$h = 2 + 20t - 5t^2$$

By completing the square, find the greatest height the ball reaches and the time when it reaches this height.

You must show all your working.

Write the terms in order: $h = -5t^2 + 20t + 2$

Take out the factor -5 from the t^2 and t terms:

$$h = -5(t^2 - 4t) + 2$$

Complete the square: $t^2 - 4t = (t - 2)^2 - 4$

$$\text{So } h = -5[(t - 2)^2 - 4] + 2$$

$$= -5(t - 2)^2 + 20 + 2$$

$$= 22 - 5(t - 2)^2$$

$5(t - 2)^2$ is never negative, so the greatest value of h is 22, when $t = 2$.

Answer: Greatest height 22 m, after 2 seconds

Tip: Take out the -5 **before** completing the square, and check the sign: $-5 \times (-4) = +20$.

Marks: 1 mark for $-5(t^2 - 4t)$. 1 mark for $-5[(t - 2)^2 - 4]$. 1 mark for $22 - 5(t - 2)^2$. 1 mark for 22 m at $t = 2$.

8 (5 marks)

- (a) Write $x^2 + 2x - 4$ in the form $(x + a)^2 + b$

Halve 2 to get 1.

$$(x + 1)^2 = x^2 + 2x + 1$$

$$x^2 + 2x - 4 = (x + 1)^2 - 1 - 4 = (x + 1)^2 - 5$$

Answer: $(x + 1)^2 - 5$

Marks: 1 mark for $(x + 1)^2$. 1 mark for $(x + 1)^2 - 5$.

- (b) Hence sketch the graph of $y = x^2 + 2x - 4$ on the axes below.
Show the coordinates of the turning point and give the exact x -coordinates of the points where the graph crosses the x -axis.

Turning point: from part (a), $y = (x + 1)^2 - 5$, so the turning point is $(-1, -5)$. It is a minimum.

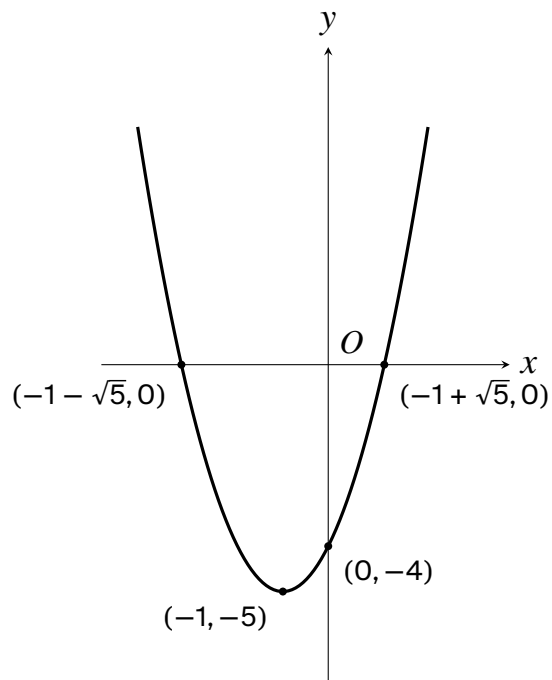
y-intercept: put $x = 0$: $y = -4$, so $(0, -4)$.

x-intercepts: put $y = 0$: $(x + 1)^2 - 5 = 0$

$$(x + 1)^2 = 5, \text{ so } x + 1 = \pm\sqrt{5}$$

$$x = -1 \pm \sqrt{5}$$

Draw a U-shaped curve through these points:



Tip: “Exact” means leave the answers as surds, $-1 \pm \sqrt{5}$. Do not change them to decimals. Label every point on your sketch.

Marks: 1 mark for the turning point $(-1, -5)$. 1 mark for $x = -1 \pm \sqrt{5}$. 1 mark for a U-shaped sketch with its minimum below the x -axis and to the left of the y -axis, crossing the y -axis below O .
