

Special Sequences - Higher

Solutions

For Pearson Edexcel GCSE Mathematics (1MA1) • Higher tier

Each question is shown in grey, followed by the working, one step per line. The final answer is in a box. *Marks* shows how the marks are usually given in the exam, and **Tip** points out a common mistake.

1 (1 mark)

The n th term of a sequence is $3^n - 1$
Find the 4th term.

Substitute $n = 4$:

$$3^4 - 1 = 81 - 1 = 80$$

Answer: 80

2 (3 marks)

Here are the first five terms of a Fibonacci sequence.

1 4 5 9 14

In a Fibonacci sequence, each term after the first two is the sum of the two terms before it.

(a) Find the 8th term of this sequence.

Keep adding the previous two terms:

$$9 + 14 = 23, \quad 14 + 23 = 37, \quad 23 + 37 = 60$$

So the sequence is 1, 4, 5, 9, 14, 23, 37, 60

Answer: 60

- (b) The first two terms of a different Fibonacci sequence are x and y .
Show that the sum of the first six terms of this sequence is 4 times the 5th term.

Write each term in terms of x and y :

$$x, y, x + y, x + 2y, 2x + 3y, 3x + 5y$$

$$\text{Sum of the first six terms: } x + y + (x + y) + (x + 2y) + (2x + 3y) + (3x + 5y) = 8x + 12y$$

$$8x + 12y = 4(2x + 3y)$$

$2x + 3y$ is the 5th term, so the sum is 4 times the 5th term.

Answer: Sum = $8x + 12y = 4(2x + 3y) = 4 \times 5\text{th term}$

Tip: Add the two terms just before each time, not the first two terms. Use brackets so you collect the x s and y s correctly.

Marks: 1 mark for the terms up to the 6th in terms of x and y . 1 mark for $8x + 12y = 4(2x + 3y)$ with a conclusion.

3 (2 marks)

The first four terms of a geometric sequence are

$$2 \quad 2\sqrt{3} \quad 6 \quad 6\sqrt{3}$$

Find the 7th term.

$$\text{Common ratio} = 2\sqrt{3} \div 2 = \sqrt{3}$$

$$\text{5th term: } 6\sqrt{3} \times \sqrt{3} = 6 \times 3 = 18$$

$$\text{6th term: } 18 \times \sqrt{3} = 18\sqrt{3}$$

$$\text{7th term: } 18\sqrt{3} \times \sqrt{3} = 18 \times 3 = 54$$

Answer: 54

Tip: Remember $\sqrt{3} \times \sqrt{3} = 3$. Multiply only the $\sqrt{3}$ parts together.

Marks: 1 mark for the common ratio $\sqrt{3}$. 1 mark for 54.

4 (3 marks)

A sequence has terms $u_1, u_2, u_3, \dots$ with $u_{n+2} = u_{n+1} + u_n$
 $u_2 = 3$ and $u_5 = 13$
Find u_1 .

The rule says: each term is the sum of the two terms before it.

$$u_3 = u_2 + u_1 = u_1 + 3$$

$$u_4 = u_3 + u_2 = (u_1 + 3) + 3 = u_1 + 6$$

$$u_5 = u_4 + u_3 = (u_1 + 6) + (u_1 + 3) = 2u_1 + 9$$

$$\text{So } 2u_1 + 9 = 13$$

$$2u_1 = 4, \text{ so } u_1 = 2$$

Check: 2, 3, 5, 8, 13 ✓

Answer: $u_1 = 2$

Marks: 1 mark for u_3 and u_4 in terms of u_1 . 1 mark for $2u_1 + 9 = 13$. 1 mark for $u_1 = 2$.

5 (3 marks)

Here are the first four terms of a Fibonacci sequence.

$$5 \quad x \quad y \quad 29$$

In a Fibonacci sequence, each term after the first two is the sum of the two terms before it.

Find the value of x and the value of y .

$$\text{3rd term: } 5 + x = y$$

$$\text{4th term: } x + y = 29$$

$$\text{Substitute } y = 5 + x: x + (5 + x) = 29$$

$$2x + 5 = 29, \text{ so } 2x = 24 \text{ and } x = 12$$

$$y = 5 + 12 = 17$$

Check: 5, 12, 17, 29 and $12 + 17 = 29$ ✓

Answer: $x = 12, y = 17$

Tip: Write the 4th term using x only: $x + (5 + x) = 2x + 5$. Then solve.

Marks: 1 mark for $y = 5 + x$ or $x + y = 29$. 1 mark for $2x + 5 = 29$. 1 mark for both values.

6 (4 marks)

Here are the first four terms of a geometric sequence.

$$81 \quad 27\sqrt{3} \quad 27 \quad 9\sqrt{3}$$

(a) Work out the 7th term of the sequence.

$$\text{Common ratio} = \frac{27\sqrt{3}}{81} = \frac{\sqrt{3}}{3}$$

$$(\text{Check: } 27\sqrt{3} \times \frac{\sqrt{3}}{3} = \frac{27 \times 3}{3} = 27 \text{ } \checkmark)$$

$$\text{5th term: } 9\sqrt{3} \times \frac{\sqrt{3}}{3} = \frac{27}{3} = 9$$

$$\text{6th term: } 9 \times \frac{\sqrt{3}}{3} = 3\sqrt{3}$$

$$\text{7th term: } 3\sqrt{3} \times \frac{\sqrt{3}}{3} = 3$$

Answer: 3

Marks: 1 mark for the common ratio $\frac{\sqrt{3}}{3}$ (or $\frac{1}{\sqrt{3}}$). 1 mark for 3.

(b) The 3rd term of a different geometric sequence is 2

The 5th term of this sequence is 6

All the terms of this sequence are positive.

Work out the 8th term of this sequence.

Give your answer in the form $a\sqrt{3}$ where a is an integer.

You must show all your working.

Let the common ratio be r .

From the 3rd term to the 5th term you multiply by r^2 .

$$r^2 = 6 \div 2 = 3$$

The terms are positive, so $r = \sqrt{3}$

From the 5th term to the 8th term you multiply by $r^3 = (\sqrt{3})^3 = 3\sqrt{3}$

$$\text{8th term} = 6 \times 3\sqrt{3} = 18\sqrt{3}$$

Answer: $18\sqrt{3}$

Tip: $(\sqrt{3})^3 = \sqrt{3} \times \sqrt{3} \times \sqrt{3} = 3\sqrt{3}$, not 9 or 3.

Marks: 1 mark for $r^2 = 3$ or $r = \sqrt{3}$. 1 mark for $18\sqrt{3}$.

7 (3 marks)

Here are the first two terms of a geometric sequence.

$$3 - \sqrt{2} \quad 7 \quad \dots$$

Work out the next term.

Give your answer in the form $a + b\sqrt{2}$ where a and b are integers.

You must show all your working.

Common ratio = 2nd term $\div$ 1st term

$$r = \frac{7}{3 - \sqrt{2}}$$

Rationalise the denominator: multiply top and bottom by $3 + \sqrt{2}$.

$$\text{Bottom: } (3 - \sqrt{2})(3 + \sqrt{2}) = 9 - 2 = 7$$

$$\text{So } r = \frac{7(3 + \sqrt{2})}{7} = 3 + \sqrt{2}$$

$$\text{3rd term} = 7 \times (3 + \sqrt{2}) = 21 + 7\sqrt{2}$$

Check: $(3 - \sqrt{2})(3 + \sqrt{2}) = 7$, so the 2nd term is right ✓

Answer: $21 + 7\sqrt{2}$

Tip: Divide a term by the term **before** it to get the ratio. To rationalise $\frac{7}{3-\sqrt{2}}$, multiply top and bottom by $3 + \sqrt{2}$.

Marks: 1 mark for $\frac{7}{3-\sqrt{2}}$. 1 mark for the common ratio $3 + \sqrt{2}$. 1 mark for $21 + 7\sqrt{2}$.

8 (4 marks)

Here are the first three terms of a geometric sequence.

$$2 \quad x \quad 3x + 8$$

All the terms of the sequence are positive.

Work out the 4th term of the sequence.

In a geometric sequence, the ratio between terms is always the same.

$$\text{So } \frac{x}{2} = \frac{3x + 8}{x}$$

$$\text{Cross-multiply: } x^2 = 2(3x + 8)$$

$$\text{Expand and rearrange: } x^2 - 6x - 16 = 0$$

$$\text{Factorise: } (x - 8)(x + 2) = 0$$

$$\text{So } x = 8 \text{ or } x = -2$$

The terms are all positive, so $x = 8$

The sequence is 2, 8, 32, so the common ratio is 4

$$\text{4th term} = 32 \times 4 = 128$$

Answer: 128

Tip: Reject $x = -2$: it gives the terms 2, -2, 2, which are not all positive.

Marks: 1 mark for setting two ratios equal. 1 mark for $x^2 - 6x - 16 = 0$. 1 mark for $x = 8$. 1 mark for 128.

9 (4 marks)

A ball is dropped from a height of 250 cm.

After each bounce, the ball reaches a height that is 60% of its previous height.

- (a) Show that the heights the ball reaches after each bounce form a geometric sequence.

Each height is 60% of the one before, so you multiply by 0.6 each time.

After bounce 1: $250 \times 0.6 = 150$ cm

After bounce 2: $150 \times 0.6 = 90$ cm

After bounce 3: $90 \times 0.6 = 54$ cm

Each term is the one before multiplied by 0.6.

There is a common ratio of 0.6, so the heights form a geometric sequence.

Answer: Each height is the one before multiplied by 0.6, so there is a common ratio of 0.6.

Marks: 1 mark for the multiplier 0.6. 1 mark for stating there is a common ratio.

- (b) After which bounce does the ball first reach a height of less than 20 cm?
You must show all your working.

Keep multiplying by 0.6:

Bounce 3: 54 cm

Bounce 4: $54 \times 0.6 = 32.4$ cm

Bounce 5: $32.4 \times 0.6 = 19.44$ cm, which is less than 20 cm

(Or: height after bounce n is 250×0.6^n , and $250 \times 0.6^4 = 32.4$,
 $250 \times 0.6^5 = 19.44$)

Answer: The 5th bounce

Tip: Show the heights either side of 20 cm so the examiner can see why the answer is the 5th bounce.

Marks: 1 mark for working out heights up to bounce 4 or 5 (32.4 and 19.44). 1 mark for the 5th bounce.

10 (3 marks)

The n th triangular number is $\frac{n(n+1)}{2}$

Prove that 9 times any triangular number, plus 1, is always a triangular number.

9 times the n th triangular number, plus 1, is $9 \times \frac{n(n+1)}{2} + 1$

$$= \frac{9n^2 + 9n}{2} + \frac{2}{2} = \frac{9n^2 + 9n + 2}{2}$$

Factorise the top: $9n^2 + 9n + 2 = (3n + 1)(3n + 2)$

So the expression is $\frac{(3n + 1)(3n + 2)}{2}$

Let $m = 3n + 1$. Then this is $\frac{m(m+1)}{2}$, the m th triangular number.

$3n + 1$ is a whole number, so the result is always a triangular number.

Answer: $\frac{(3n + 1)(3n + 2)}{2}$, the $(3n + 1)$ th triangular number.

Tip: To show a number is triangular, get it into the form $\frac{m(m+1)}{2}$: two consecutive whole numbers multiplied, then halved.

Marks: 1 mark for $\frac{9n^2+9n+2}{2}$. 1 mark for $(3n + 1)(3n + 2)$. 1 mark for linking this to $\frac{m(m+1)}{2}$ with a conclusion.