

Iteration

Solutions

For Pearson Edexcel GCSE Mathematics (1MA1) • Higher tier

Each question is shown in grey, followed by the working, one step per line. The final answer is in a box. *Marks* shows how the marks are usually given in the exam, and **Tip** points out a common mistake.

1 (3 marks)

Using $x_{n+1} = \sqrt{3x_n + 5}$ with $x_0 = 4$, find the values of x_1 , x_2 and x_3
Give your answers correct to 3 decimal places.

Put each answer back into the formula to get the next one.

$$x_1 = \sqrt{3 \times 4 + 5} = \sqrt{17} = 4.1231 \dots$$

$$x_2 = \sqrt{3 \times 4.1231 \dots + 5} = \sqrt{17.369 \dots} = 4.1676 \dots$$

$$x_3 = \sqrt{3 \times 4.1676 \dots + 5} = \sqrt{17.502 \dots} = 4.1836 \dots$$

Round each value to 3 decimal places.

Answer: $x_1 = 4.123$, $x_2 = 4.168$, $x_3 = 4.184$

Tip: Each new value comes from the value **before** it. Do not substitute 0, 1, 2, ... into the formula. Use the ANS key so you always use the full, unrounded value.

Marks: 1 mark for substituting $x_0 = 4$ correctly. 1 mark for x_1 and x_2 . 1 mark for x_3 .

2 (9 marks)

(a) Show that the equation $x^3 + x^2 - 6 = 0$ has a solution between 1 and 2

Substitute $x = 1$ and $x = 2$ into $x^3 + x^2 - 6$.

When $x = 1$: $1 + 1 - 6 = -4$ (negative)

When $x = 2$: $8 + 4 - 6 = 6$ (positive)

The value changes sign, so there is a solution between 1 and 2.

Answer: -4 and 6: there is a change of sign, so there is a solution between 1 and 2.

Tip: Your conclusion must mention the **change of sign**. Examiners often saw correct values with no statement, which loses the second mark.

Marks: 1 mark for working out both values. 1 mark for saying there is a change of sign, so there is a solution.

- (b) Show that the equation $x^3 + x^2 - 6 = 0$ can be rearranged to give $x = \sqrt{\frac{6}{x+1}}$

Add 6 to both sides: $x^3 + x^2 = 6$

Factorise the left side: $x^2(x+1) = 6$

Divide both sides by $(x+1)$: $x^2 = \frac{6}{x+1}$

Square root both sides: $x = \sqrt{\frac{6}{x+1}}$

Answer: $x = \sqrt{\frac{6}{x+1}}$

Tip: Look for a common factor: $x^3 + x^2 = x^2(x+1)$. Then only x^2 is left on one side.

Marks: 1 mark for $x^2(x+1) = 6$. 1 mark for $x^2 = \frac{6}{x+1}$ leading to the given form.

- (c) Starting with $x_0 = 1.5$, use the iteration formula $x_{n+1} = \sqrt{\frac{6}{x_n+1}}$ three times to find an estimate for a solution of $x^3 + x^2 - 6 = 0$

$$x_1 = \sqrt{\frac{6}{1.5+1}} = \sqrt{2.4} = 1.54919 \dots$$

$$x_2 = \sqrt{\frac{6}{1.54919 \dots + 1}} = 1.53417 \dots$$

$$x_3 = \sqrt{\frac{6}{1.53417 \dots + 1}} = 1.53871 \dots$$

Answer: 1.5387 (1.53871...)

Tip: “Three times” means stop at x_3 . Going further, or averaging the values, is not what the question asks.

Marks: 1 mark for $x_1 = 1.549 \dots$ 1 mark for x_2 . 1 mark for x_3 .

- (d) By substituting your answer to part (c) into $x^3 + x^2 - 6$, comment on the accuracy of your estimate for the solution to $x^3 + x^2 - 6 = 0$

$$1.5387^3 + 1.5387^2 - 6 = 0.0106 \dots$$

This is very close to 0.

So 1.5387 is a good estimate of the solution (it is accurate to about 2 decimal places).

Answer: The value is close to 0, so it is a good estimate.

Marks: 1 mark for substituting your estimate. 1 mark for saying the value is close to 0, so it is a good estimate.

3 (4 marks)

- (a) Show that the equation $x^3 - 4x - 9 = 0$ can be written in the form $x = \sqrt{4 + \frac{9}{x}}$

Add $4x$ and 9 to both sides: $x^3 = 4x + 9$

Divide every term by x : $x^2 = 4 + \frac{9}{x}$

Square root both sides: $x = \sqrt{4 + \frac{9}{x}}$

Answer: $x = \sqrt{4 + \frac{9}{x}}$

Tip: Write down the middle steps $x^3 = 4x + 9$ and $x^2 = 4 + \frac{9}{x}$. In “show that” questions you must show every step.

(b) Starting with $x_0 = 3$

use the iteration formula $x_{n+1} = \sqrt{4 + \frac{9}{x_n}}$ three times to find an estimate for a solution of $x^3 - 4x - 9 = 0$

$$x_1 = \sqrt{4 + \frac{9}{3}} = \sqrt{7} = 2.6457 \dots$$

$$x_2 = \sqrt{4 + \frac{9}{2.6457 \dots}} = 2.7206 \dots$$

$$x_3 = \sqrt{4 + \frac{9}{2.7206 \dots}} = 2.7033 \dots$$

Answer: 2.703 (2.7033...)

Tip: On your calculator, make sure the square root covers the whole of $4 + \frac{9}{x_n}$.

Marks: 1 mark for the first substitution. 1 mark for x_2 . 1 mark for x_3 .

4 (4 marks)

(a) Use the iteration formula $x_{n+1} = 4 - \frac{1}{x_n^2}$ to find the values of x_1 , x_2 and x_3
Start with $x_0 = 4$

$$x_1 = 4 - \frac{1}{4^2} = 4 - 0.0625 = 3.9375$$

$$x_2 = 4 - \frac{1}{3.9375^2} = 3.93550 \dots$$

$$x_3 = 4 - \frac{1}{3.93550 \dots^2} = 3.93543 \dots$$

Answer: $x_1 = 3.9375$, $x_2 = 3.93550 \dots$, $x_3 = 3.93543 \dots$

Marks: 1 mark for the first substitution. 1 mark for x_1 and x_2 . 1 mark for x_3 .

(b) The values of x_1 , x_2 and x_3 found in part (a) are estimates of a solution of an equation of the form $x^3 + px^2 + q = 0$ where p and q are integers.
Find the value of p and the value of q .

The iteration formula comes from the equation $x = 4 - \frac{1}{x^2}$

Multiply every term by x^2 : $x^3 = 4x^2 - 1$

Rearrange so one side is 0: $x^3 - 4x^2 + 1 = 0$

Compare with $x^3 + px^2 + q = 0$.

Answer: $p = -4$, $q = 1$

Tip: Multiply **every** term by x^2 , including the 4.

5 (6 marks)

- (a) Show that the equation $x^3 - 3x^2 - 2 = 0$ has a solution between $x = 3$ and $x = 4$

Substitute $x = 3$ and $x = 4$ into $x^3 - 3x^2 - 2$.

When $x = 3$: $27 - 27 - 2 = -2$ (negative)

When $x = 4$: $64 - 48 - 2 = 14$ (positive)

The value changes sign, so there is a solution between 3 and 4.

Answer: -2 and 14 : there is a change of sign, so there is a solution between 3 and 4.

Marks: 1 mark for working out both values. 1 mark for saying there is a change of sign, so there is a solution.

- (b) Show that the equation $x^3 - 3x^2 - 2 = 0$ can be rearranged to give $x = \sqrt[3]{3x^2 + 2}$

Add $3x^2$ and 2 to both sides: $x^3 = 3x^2 + 2$

Take the cube root of both sides: $x = \sqrt[3]{3x^2 + 2}$

Answer: $x = \sqrt[3]{3x^2 + 2}$

- (c) Starting with $x_0 = 3.2$

use the iteration formula $x_{n+1} = \sqrt[3]{3x_n^2 + 2}$ twice to find an estimate for the solution of $x^3 - 3x^2 - 2 = 0$

Give your answer correct to 4 decimal places.

$$x_1 = \sqrt[3]{3 \times 3.2^2 + 2} = \sqrt[3]{32.72} = 3.19843 \dots$$

$$x_2 = \sqrt[3]{3 \times 3.19843 \dots^2 + 2} = 3.19745 \dots$$

“Twice” means stop at x_2 .

Round x_2 to 4 decimal places.

Answer: 3.1975

Tip: On your calculator, make sure the cube root covers the whole of $3x_n^2 + 2$.

Marks: 1 mark for the first substitution. 1 mark for x_1 . 1 mark for x_2 correct to 4 decimal places.
