

Congruent Triangles and Geometric Proof

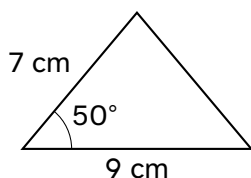
Solutions

For Pearson Edexcel GCSE Mathematics (1MA1) • Higher tier

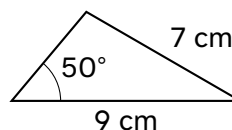
Each question is shown in grey, followed by the working, one step per line. The final answer is in a box. *Marks* shows how the marks are usually given in the exam, and **Tip** points out a common mistake.

1 (2 marks)

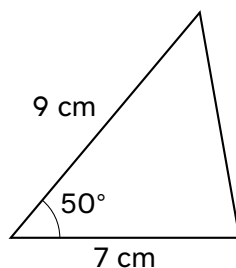
Here are four triangles, X , P , Q and R .



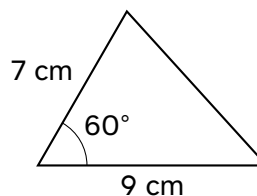
Triangle X



Triangle P



Triangle Q



Triangle R

Diagram **NOT** accurately drawn

One of the triangles P , Q and R is congruent to triangle X .

Which one?

Give a reason for your answer.

Triangle X has sides of 7 cm and 9 cm with the 50° angle between them.

Triangle Q also has sides of 7 cm and 9 cm with the 50° angle between them.

So X and Q are congruent (SAS).

In triangle P the 50° angle is not between the 7 cm and 9 cm sides.

Triangle R has a 60° angle, not 50° .

Answer: Q (two sides and the included angle are equal: SAS)

Tip: Check where the angle is. For SAS the equal angle must be **between** the two equal sides.

Marks: 1 mark for Q . 1 mark for a reason: two pairs of equal sides and the angle between them are equal (SAS).

2 (3 marks)

Here is triangle ABC .

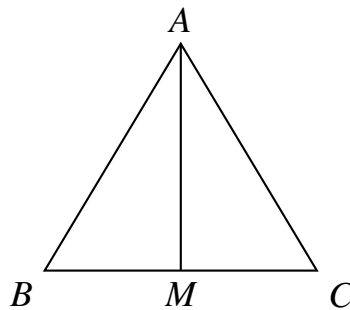


Diagram **NOT** accurately drawn

ABC is an isosceles triangle with $AB = AC$.

M is the midpoint of BC .

Prove that triangle ABM is congruent to triangle ACM .

$AB = AC$ (given)

$BM = CM$ (M is the midpoint of BC)

AM is a side of both triangles (common side).

All three pairs of sides are equal.

So triangles ABM and ACM are congruent (SSS).

Tip: Write “(given)” after facts from the question, such as $AB = AC$. Every statement in a proof needs a reason.

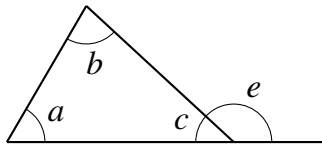
Marks: 1 mark for one pair of equal sides with a reason. 1 mark for all three pairs with reasons. 1 mark for the conclusion with SSS.

3 (3 marks)

Prove that the exterior angle of a triangle is equal to the sum of the two opposite interior angles.

Draw a triangle and extend one side. Call the interior angles a , b and c .

Call the exterior angle next to c angle e .



$$a + b + c = 180^\circ \text{ (angles in a triangle add up to } 180^\circ\text{)}$$

$$c + e = 180^\circ \text{ (angles on a straight line add up to } 180^\circ\text{)}$$

$$\text{Both equal } 180^\circ, \text{ so } a + b + c = c + e$$

$$\text{Subtract } c \text{ from both sides: } a + b = e$$

So the exterior angle equals the sum of the two opposite interior angles.

Tip: Use letters, not numbers. A proof with one example (like 50° , 60° , 70°) does not show it is always true.

Marks: 1 mark for using angles in a triangle. 1 mark for using angles on a straight line. 1 mark for a complete proof.

4 (3 marks)

$ABCD$ is a square. E is a point on BC and F is a point on DC such that $BE = CF$.

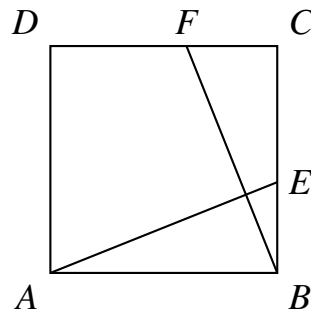


Diagram **NOT** accurately drawn

Prove that triangle ABE is congruent to triangle BCF .

$AB = BC$ (the sides of a square are equal)

Angle $ABE = \text{angle } BCF = 90^\circ$ (all angles in a square are 90°)

$BE = CF$ (given)

The equal angle is between the two pairs of equal sides.

So triangles ABE and BCF are congruent (SAS).

Marks: 1 mark for one pair of equal sides or angles with a reason. 1 mark for all three pairs with reasons. 1 mark for the conclusion with SAS.

5 (4 marks)

$ABCD$ is a parallelogram. X is the midpoint of the diagonal AC .
The straight line PXQ meets AB at P and DC at Q .

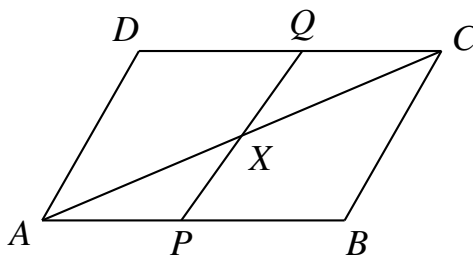


Diagram **NOT** accurately drawn

Prove that X is the midpoint of PQ .

Look at triangles APX and CQX .

$AX = CX$ (X is the midpoint of AC)

Angle $PAX =$ angle QCX (alternate angles are equal, as AB is parallel to DC)

Angle $AXP =$ angle CXQ (vertically opposite angles are equal)

So triangles APX and CQX are congruent (ASA).

So $PX = QX$ (corresponding sides of congruent triangles)

So X is the midpoint of PQ .

Tip: Give the full reasons “alternate angles are equal” and “vertically opposite angles are equal”. Just marking the equal angles on the diagram is not enough.

Marks: 1 mark for $AX = CX$ with a reason. 1 mark for two pairs of equal angles with reasons. 1 mark for congruent (ASA). 1 mark for $PX = QX$, so X is the midpoint of PQ .

6 (3 marks)

Here is a trapezium.

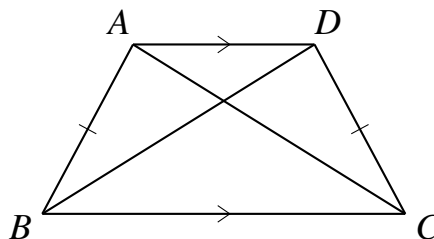


Diagram **NOT** accurately drawn

$ABCD$ is a trapezium with AD parallel to BC .

$AB = DC$ and angle $ABC = \text{angle } DCB$.

Prove that $AC = BD$.

Look at triangles ABC and DCB .

$AB = DC$ (given)

Angle $ABC = \text{angle } DCB$ (given)

BC is a side of both triangles (common side).

So triangles ABC and DCB are congruent (SAS).

So $AC = DB$ (corresponding sides of congruent triangles are equal)

Tip: Choose two triangles that contain the sides you need (AC and BD), then prove those triangles are congruent.

Marks: 1 mark for choosing triangles ABC and DCB with one pair of equal sides or angles. 1 mark for all three pairs. 1 mark for SAS and the conclusion $AC = BD$.

7 (3 marks)

Here is a circle, centre O .

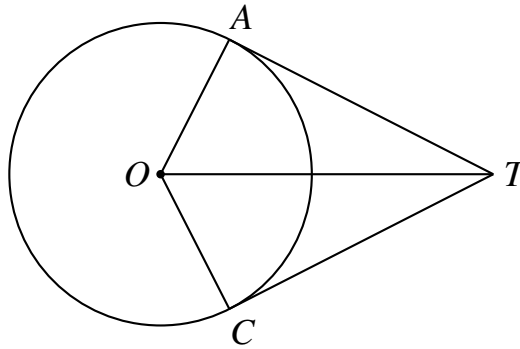


Diagram **NOT** accurately drawn

A and C are points on a circle, centre O .

TA and TC are tangents to the circle.

Prove that $TA = TC$.

Look at triangles OAT and OCT .

$OA = OC$ (radii of the same circle)

Angle $OAT = \text{angle } OCT = 90^\circ$ (a tangent is perpendicular to the radius at the point of contact)

OT is a side of both triangles (common side) and it is the hypotenuse of both.

So triangles OAT and OCT are congruent (RHS).

So $TA = TC$ (corresponding sides of congruent triangles)

Tip: You cannot use “tangents from a point are equal” as a reason here: that is what you are proving.

Marks: 1 mark for $OA = OC$ (radii) or the right angles with a reason. 1 mark for congruent (RHS) with all three facts. 1 mark for $TA = TC$ from the congruent triangles.

8 (3 marks)

The diagram shows a triangle ABC . D is a point on AB and E is a point on AC .

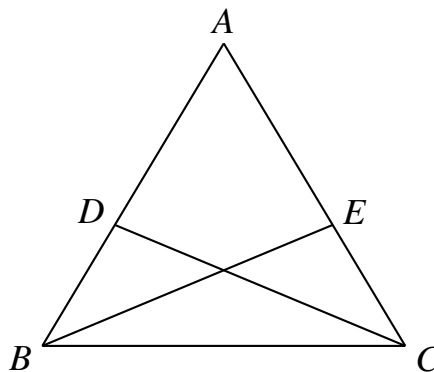


Diagram **NOT** accurately drawn

$$AB = AC$$

$$BD = CE$$

Prove that triangle DBC is congruent to triangle ECB .

$$BD = CE \text{ (given)}$$

$AB = AC$, so triangle ABC is an isosceles triangle.

Angle $DBC = \text{angle } ECB$ (base angles of an isosceles triangle are equal)

BC is a side of both triangles (common side).

The equal angles are between the equal sides.

So triangles DBC and ECB are congruent (SAS).

Tip: Write “isosceles triangle”, not just “isosceles”, and say what is equal: “base angles of an isosceles triangle are equal”.

Marks: 1 mark for angle $DBC = \text{angle } ECB$ with a reason. 1 mark for $BD = CE$ and the common side BC . 1 mark for a complete proof with SAS.

9 (5 marks)

$ABCD$ is a rectangle.

M is the midpoint of AB .

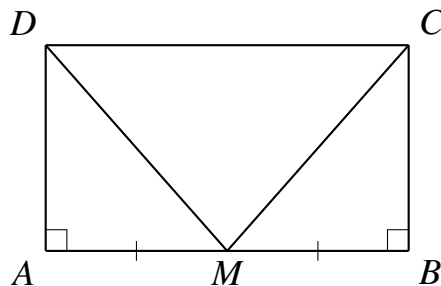


Diagram **NOT** accurately drawn

- (a) Prove that triangle AMD is congruent to triangle BMC .

$AM = BM$ (M is the midpoint of AB)

$AD = BC$ (opposite sides of a rectangle are equal)

Angle $DAM = \text{angle } CBM = 90^\circ$ (all angles in a rectangle are 90°)

The right angles are between the equal sides.

So triangles AMD and BMC are congruent (SAS).

Marks: 1 mark for one pair with a reason. 1 mark for all three pairs with reasons. 1 mark for the conclusion with SAS.

- (b) Hence show that angle $MDC = \text{angle } MCD$.

$MD = MC$ (corresponding sides of the congruent triangles in part (a))

So triangle DMC is an isosceles triangle.

So angle $MDC = \text{angle } MCD$ (base angles of an isosceles triangle are equal)

Tip: "Hence" means use your answer to part (a): the congruent triangles give $MD = MC$.

Marks: 1 mark for $MD = MC$ with a reason. 1 mark for the base angles of isosceles triangle DMC being equal.