

Algebraic Proof

Solutions

For Pearson Edexcel GCSE Mathematics (1MA1) • Higher tier

Each question is shown in grey, followed by the working, one step per line. The final answer is in a box. *Marks* shows how the marks are usually given in the exam, and **Tip** points out a common mistake.

1 (3 marks)

n is an integer.

Write down, in terms of n , an expression for

(a) any odd number

$2n$ is always even, because it is a multiple of 2.

One more than an even number is odd.

Answer: $2n + 1$

(b) the next odd number after $2n + 1$

Odd numbers go up in 2s.

$$2n + 1 + 2 = 2n + 3$$

Answer: $2n + 3$

(c) the sum of the two consecutive integers n and $n + 1$

$$n + (n + 1) = 2n + 1$$

Answer: $2n + 1$

2 (2 marks)

Prove algebraically that the sum of two consecutive odd numbers is always a multiple of 4

Let the two consecutive odd numbers be $2n + 1$ and $2n + 3$, where n is an integer.

Sum: $(2n + 1) + (2n + 3) = 4n + 4$

Factorise: $4n + 4 = 4(n + 1)$

$n + 1$ is an integer, so $4(n + 1)$ is a multiple of 4.

Answer: $4(n + 1)$

Tip: Don't use n and $n + 2$ for odd numbers, because n could be even. $2n + 1$ is always odd.

Marks: 1 mark for $(2n + 1) + (2n + 3)$. 1 mark for $4(n + 1)$ with a conclusion.

3 (3 marks)

Prove algebraically that the sum of any two consecutive even numbers is not a multiple of 4

Let the two consecutive even numbers be $2n$ and $2n + 2$, where n is an integer.

Sum: $2n + (2n + 2) = 4n + 2$

$4n$ is a multiple of 4.

So $4n + 2$ is 2 more than a multiple of 4.

It leaves a remainder of 2 when divided by 4, so it is not a multiple of 4.

Answer: $4n + 2$

Tip: Writing $4n + 2 = 2(2n + 1)$ only shows the sum is even. You must explain why it is **not** a multiple of 4.

Marks: 1 mark for $2n + (2n + 2)$. 1 mark for $4n + 2$. 1 mark for a conclusion explaining why it is not a multiple of 4.

4 (3 marks)

n is an integer.

Prove algebraically that $(3n + 1)^2 + (n - 3)^2$ is always a multiple of 10

Expand the first bracket: $(3n + 1)^2 = 9n^2 + 6n + 1$

Expand the second bracket: $(n - 3)^2 = n^2 - 6n + 9$

Add: $9n^2 + 6n + 1 + n^2 - 6n + 9$

$= 10n^2 + 10$

Take out the common factor 10: $= 10(n^2 + 1)$

$n^2 + 1$ is an integer, so this is a multiple of 10.

Answer: $10(n^2 + 1)$

Tip: $(3n + 1)^2$ is not $9n^2 + 1$. Write it as $(3n + 1)(3n + 1)$ and expand fully to get the middle term $6n$.

Marks: 1 mark for expanding both brackets. 1 mark for $10n^2 + 10$. 1 mark for $10(n^2 + 1)$ with a conclusion.

5 (3 marks)

n is a positive integer.

Prove algebraically that $(3n + 2)(n + 4) - n(n + 6)$ is always an even number.

Expand: $(3n + 2)(n + 4) = 3n^2 + 12n + 2n + 8 = 3n^2 + 14n + 8$

Expand: $n(n + 6) = n^2 + 6n$

Subtract, keeping the brackets: $3n^2 + 14n + 8 - (n^2 + 6n)$

$= 2n^2 + 8n + 8$

Take out a factor of 2: $= 2(n^2 + 4n + 4)$

This is $2 \times$ an integer, so it is always even.

Answer: $2(n^2 + 4n + 4)$

Tip: Stopping at $2n^2 + 8n + 8$ is not enough. Take out the factor 2 and say that $2 \times$ an integer is even.

Marks: 1 mark for expanding both brackets correctly. 1 mark for $2n^2 + 8n + 8$. 1 mark for $2(n^2 + 4n + 4)$ with a conclusion.

6 (3 marks)

Two integers have a difference of 1

Prove algebraically that the square of their sum is always 1 more than 4 times their product.

Let the integers be n and $n + 1$.

Their sum is $2n + 1$.

Square of their sum: $(2n + 1)^2 = 4n^2 + 4n + 1$

4 times their product: $4n(n + 1) = 4n^2 + 4n$

So $(2n + 1)^2 = (4n^2 + 4n) + 1$.

The square of their sum is 1 more than 4 times their product.

Answer: $(2n + 1)^2 = 4n(n + 1) + 1$

Tip: Square the **whole** sum: $(2n + 1)^2 = 4n^2 + 4n + 1$, not $2n^2 + 1$ or $4n^2 + 1$.

Marks: 1 mark for $(2n + 1)^2$ or $4n(n + 1)$. 1 mark for $4n^2 + 4n + 1$. 1 mark for comparing it with $4n^2 + 4n$.

7 (3 marks)

n is a positive integer.

Prove that $\frac{n^2 + 5n + 6}{n + 2} + \frac{n^2 - 1}{n + 1}$ is always an even number.

Factorise the first numerator: $n^2 + 5n + 6 = (n + 2)(n + 3)$

$$\text{So } \frac{(n+2)(n+3)}{n+2} = n+3$$

Factorise the second numerator (difference of two squares): $n^2 - 1 = (n+1)(n-1)$

$$\text{So } \frac{(n+1)(n-1)}{n+1} = n-1$$

$$\text{Add: } (n+3) + (n-1) = 2n+2$$

$2n+2 = 2(n+1)$, which is $2 \times$ an integer, so it is always even.

Answer: $2(n+1)$

Tip: You can only cancel a whole factor. Factorise the top first; don't cancel the n in n^2 with the n in $n+2$.

Marks: 1 mark for factorising one numerator correctly. 1 mark for $n+3$ and $n-1$. 1 mark for $2(n+1)$ with a conclusion.

8 (2 marks)

n is an integer.

Prove that $n^2 + 5n + 3$ is always an odd number.

Write it as $n(n+5) + 3$

If n is even, then $n(n+5)$ is even $\times$ an integer, so it is even.

If n is odd, then $n+5$ is even, so $n(n+5)$ is even.

Either way $n(n+5)$ is even, and even $+ 3$ is odd.

So $n^2 + 5n + 3$ is always odd.

Answer: $n(n+5) + 3$ is always odd

Tip: Check both cases: n even and n odd. Covering only one case is not a full proof.

Marks: 1 mark for $n(n+5) + 3$ (or looking at n odd and n even). 1 mark for a complete argument.

9 (3 marks)

n is an integer greater than 1

Prove that $n^3 - n$ is always a multiple of 6

Take out the common factor n : $n^3 - n = n(n^2 - 1)$

Use the difference of two squares: $= n(n - 1)(n + 1)$

Reorder: $(n - 1) \times n \times (n + 1)$ is the product of three consecutive integers.

Of any two consecutive integers, one is even, so the product is a multiple of 2.

Of any three consecutive integers, one is a multiple of 3, so the product is a multiple of 3.

A multiple of both 2 and 3 is a multiple of 6.

Answer: $(n - 1)n(n + 1)$

Tip: Trying a few values of n is not a proof. Factorise fully, then explain why the product must be a multiple of both 2 and 3.

Marks: 1 mark for $n(n^2 - 1)$. 1 mark for $(n - 1)n(n + 1)$. 1 mark for explaining why three consecutive integers give a multiple of 2 and of 3.

10 (5 marks)

(a) Prove that

$$(n - 1)^2 + n^2 + (n + 1)^2 + (n + 2)^2 = 4(n^2 + n + 1) + 2$$

Expand: $(n - 1)^2 = n^2 - 2n + 1$

Expand: $(n + 1)^2 = n^2 + 2n + 1$

Expand: $(n + 2)^2 = n^2 + 4n + 4$

Add all four: $n^2 - 2n + 1 + n^2 + n^2 + 2n + 1 + n^2 + 4n + 4$
 $= 4n^2 + 4n + 6$

Take out 4 from the first two terms and write $6 = 4 + 2$: $= 4(n^2 + n + 1) + 2$ ✓

Answer: Shown

Tip: $(n - 1)^2$ is $n^2 - 2n + 1$, not $n^2 - 1$ or $n^2 + 1$. Write it as $(n - 1)(n - 1)$ to get all the terms.

Marks: 1 mark for expanding one bracket correctly. 1 mark for $4n^2 + 4n + 6$. 1 mark for a fully correct proof.

(b) n is an integer.

Ella says, “The sum of the squares of four consecutive integers can never be a multiple of 4”

Use your answer to part (a) to explain why Ella is correct.

Four consecutive integers can be written as $n - 1$, n , $n + 1$ and $n + 2$.

From part (a), the sum of their squares is $4(n^2 + n + 1) + 2$.

$n^2 + n + 1$ is an integer, so $4(n^2 + n + 1)$ is a multiple of 4.

So the sum of the squares is 2 more than a multiple of 4.

It leaves a remainder of 2 when divided by 4, so it can never be a multiple of 4.

Answer: $4(n^2 + n + 1) + 2$ is 2 more than a multiple of 4, so it is never a multiple of 4

Tip: Numerical examples are not enough here. You need a reason that works for any four consecutive integers.

Marks: 1 mark for saying $4(n^2 + n + 1)$ is a multiple of 4. 1 mark for a complete reason (2 more than a multiple of 4, so not a multiple of 4).