

Vector Proof

Solutions

For AQA GCSE Mathematics (8300) • Higher tier

Each question is shown in grey, followed by the working, one step per line. The final answer is in a box. *Marks* shows how the marks are usually given in the exam, and **Tip** points out a common mistake.

1 (1 mark)

Which vector is parallel to $2\mathbf{a} - 3\mathbf{b}$?

Circle your answer.

Options: $3\mathbf{a} - 2\mathbf{b}$, $4\mathbf{a} + 6\mathbf{b}$, $6\mathbf{a} - 9\mathbf{b}$, $2\mathbf{a} + 3\mathbf{b}$

Parallel vectors are multiples of each other.

$$6\mathbf{a} - 9\mathbf{b} = 3(2\mathbf{a} - 3\mathbf{b})$$

The others are not multiples: for example, $4\mathbf{a} + 6\mathbf{b}$ has the wrong sign on $\mathbf{b}$.

Answer: $6\mathbf{a} - 9\mathbf{b}$

2 (2 marks)

$$\overrightarrow{OA} = \mathbf{a} \text{ and } \overrightarrow{OB} = \mathbf{b}$$

P is the point on AB such that $AP:PB = 2:3$

Write $\overrightarrow{OP}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

Give your answer in its simplest form.

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\mathbf{a} + \mathbf{b} = \mathbf{b} - \mathbf{a}$$

$AP:PB = 2:3$ is 5 parts, so AP is $\frac{2}{5}$ of AB .

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = \mathbf{a} + \frac{2}{5}(\mathbf{b} - \mathbf{a})$$

$$= \mathbf{a} + \frac{2}{5}\mathbf{b} - \frac{2}{5}\mathbf{a}$$

$$= \frac{3}{5}\mathbf{a} + \frac{2}{5}\mathbf{b}$$

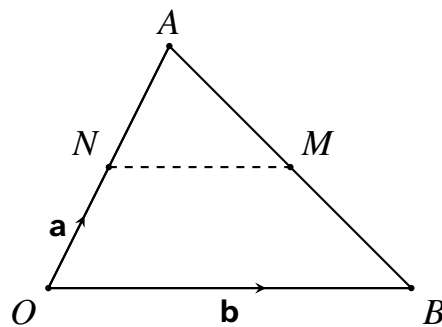
Answer: $\frac{3}{5}\mathbf{a} + \frac{2}{5}\mathbf{b}$

Tip: $AP:PB = 2:3$ means AP is $\frac{2}{5}$ of AB , not $\frac{2}{3}$. Add the parts of the ratio first.

Marks: 1 mark for $\mathbf{a} + \frac{2}{5}(\mathbf{b} - \mathbf{a})$. 1 mark for $\frac{3}{5}\mathbf{a} + \frac{2}{5}\mathbf{b}$.

3 (6 marks)

$\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$



Not drawn accurately

(a) Write $\overrightarrow{AB}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

Go from A to O , then from O to B .

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\mathbf{a} + \mathbf{b}$$

Answer: $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

Tip: Going against the arrow for $\mathbf{a}$ gives $-\mathbf{a}$. The answer is $\mathbf{b} - \mathbf{a}$, not $\mathbf{a} - \mathbf{b}$.

(b) M is the midpoint of AB .

Write $\overrightarrow{OM}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

$$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$$

M is the midpoint, so $\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}(\mathbf{b} - \mathbf{a})$

$$\overrightarrow{OM} = \mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a})$$

$$= \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$$

Answer: $\overrightarrow{OM} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$

Marks: 1 mark for $\mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a})$. 1 mark for $\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$.

- (c) N is the midpoint of OA .
Prove that NM is parallel to OB and half its length.

N is the midpoint of OA , so $\overrightarrow{NO} = -\frac{1}{2}\mathbf{a}$

$$\overrightarrow{NM} = \overrightarrow{NO} + \overrightarrow{OM}$$

$$= -\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$$

$$= \frac{1}{2}\mathbf{b}$$

$$\overrightarrow{OB} = \mathbf{b}, \text{ so } \overrightarrow{NM} = \frac{1}{2}\overrightarrow{OB}$$

So NM is parallel to OB and half its length.

Marks: 1 mark for a correct route for $\overrightarrow{NM}$. 1 mark for $\frac{1}{2}\mathbf{b}$. 1 mark for the conclusion.

4 (2 marks)

$$\overrightarrow{PQ} = 3\mathbf{a} - 2\mathbf{b} \text{ and } \overrightarrow{QR} = 6\mathbf{a} - 4\mathbf{b}$$

Prove that P , Q and R lie on a straight line.

$$6\mathbf{a} - 4\mathbf{b} = 2(3\mathbf{a} - 2\mathbf{b})$$

$$\text{So } \overrightarrow{QR} = 2\overrightarrow{PQ}$$

This means PQ and QR are parallel.

They also share the point Q .

So P , Q and R lie on a straight line.

Tip: You need two things: the vectors are multiples (parallel) **and** they share a point. Showing $\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$ proves nothing.

Marks: 1 mark for $\overrightarrow{QR} = 2\overrightarrow{PQ}$. 1 mark for the conclusion: parallel and a common point.

5 (2 marks)

$$\overrightarrow{OX} = 2\mathbf{a} + \mathbf{b} \text{ and } \overrightarrow{OY} = \mathbf{a} + k\mathbf{b}$$

$\overrightarrow{OY}$ is parallel to $\overrightarrow{OX}$.

Work out the value of k .

Parallel, so $\overrightarrow{OY}$ is a multiple of $\overrightarrow{OX}$.

The $\mathbf{a}$ part of $\overrightarrow{OY}$ is half the $\mathbf{a}$ part of $\overrightarrow{OX}$.

$$\text{So } \overrightarrow{OY} = \frac{1}{2}(2\mathbf{a} + \mathbf{b}) = \mathbf{a} + \frac{1}{2}\mathbf{b}$$

Compare with $\mathbf{a} + k\mathbf{b}$: $k = \frac{1}{2}$

Answer: $k = \frac{1}{2}$

Marks: 1 mark for $\overrightarrow{OY} = \frac{1}{2}\overrightarrow{OX}$. 1 mark for $\frac{1}{2}$.

6 (3 marks)

$$\overrightarrow{OA} = \mathbf{a} \text{ and } \overrightarrow{OB} = \mathbf{b}$$

$$\overrightarrow{OC} = 2\mathbf{a} + k\mathbf{b}, \text{ where } k \text{ is a number.}$$

A , B and C lie on a straight line.

Work out the value of k .

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$$

$$\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} = -\mathbf{a} + 2\mathbf{a} + k\mathbf{b} = \mathbf{a} + k\mathbf{b}$$

A , B and C are on a straight line, so $\overrightarrow{AC}$ is a multiple of $\overrightarrow{AB}$.

The $\mathbf{a}$ part of $\overrightarrow{AC}$ is $+\mathbf{a}$, and in $\overrightarrow{AB}$ it is $-\mathbf{a}$, so the multiple is -1 .

$$\overrightarrow{AC} = -1(\mathbf{b} - \mathbf{a}) = \mathbf{a} - \mathbf{b}$$

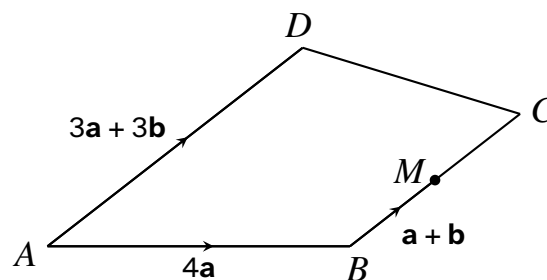
So $k = -1$

Answer: $k = -1$

Marks: 1 mark for $\overrightarrow{AB}$ or $\overrightarrow{AC}$. 1 mark for writing $\overrightarrow{AC}$ as a multiple of $\mathbf{b} - \mathbf{a}$. 1 mark for -1 .

7 (3 marks)

Here is a quadrilateral.



Not drawn accurately

$ABCD$ is a quadrilateral.

M is the midpoint of BC .

$\overrightarrow{AB} = 4\mathbf{a}$, $\overrightarrow{BM} = \mathbf{a} + \mathbf{b}$ and $\overrightarrow{AD} = 3\mathbf{a} + 3\mathbf{b}$

Prove that $ABCD$ is a trapezium.

M is the midpoint of BC , so $\overrightarrow{BC} = 2\overrightarrow{BM} = 2\mathbf{a} + 2\mathbf{b} = 2(\mathbf{a} + \mathbf{b})$

$\overrightarrow{AD} = 3\mathbf{a} + 3\mathbf{b} = 3(\mathbf{a} + \mathbf{b})$

So $\overrightarrow{AD} = \frac{3}{2}\overrightarrow{BC}$, which means AD is parallel to BC .

Check the other pair of sides: $\overrightarrow{DC} = \overrightarrow{DA} + \overrightarrow{AB} + \overrightarrow{BC} = -3\mathbf{a} - 3\mathbf{b} + 4\mathbf{a} + 2\mathbf{a} + 2\mathbf{b} = 3\mathbf{a} - \mathbf{b}$

$3\mathbf{a} - \mathbf{b}$ is not a multiple of $4\mathbf{a}$, so DC is not parallel to AB .

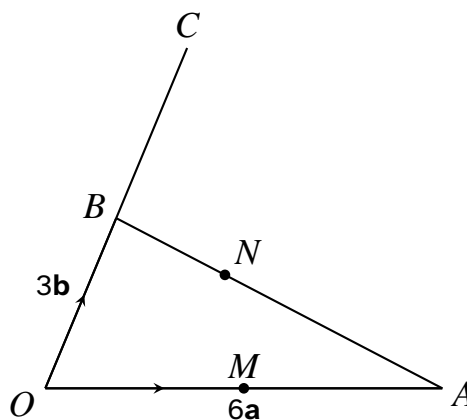
$ABCD$ has exactly one pair of parallel sides, so it is a trapezium.

Tip: A parallelogram also has parallel sides. To prove a trapezium, show one pair of sides **is** parallel and the other pair is **not**.

Marks: 1 mark for $\overrightarrow{BC} = 2\mathbf{a} + 2\mathbf{b}$. 1 mark for showing $\overrightarrow{AD}$ is a multiple of $\overrightarrow{BC}$. 1 mark for showing AB and DC are not parallel, with the conclusion.

8 (4 marks)

Here is a triangle.



Not drawn accurately

OAB is a triangle. $\overrightarrow{OA} = 6\mathbf{a}$ and $\overrightarrow{OB} = 3\mathbf{b}$

M is the midpoint of OA .

N is the point on AB such that $AN:NB = 2:1$

C is the point such that OBC is a straight line and $OB = BC$.

Prove that M , N and C lie on a straight line.

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -6\mathbf{a} + 3\mathbf{b}$$

$$\overrightarrow{AN} = \frac{2}{3}\overrightarrow{AB} = -4\mathbf{a} + 2\mathbf{b}$$

$$\overrightarrow{MN} = \overrightarrow{MA} + \overrightarrow{AN} = 3\mathbf{a} - 4\mathbf{a} + 2\mathbf{b} = -\mathbf{a} + 2\mathbf{b}$$

$$\overrightarrow{OC} = 2\overrightarrow{OB} = 6\mathbf{b}, \text{ so } \overrightarrow{MC} = \overrightarrow{MO} + \overrightarrow{OC} = -3\mathbf{a} + 6\mathbf{b}$$

$$-3\mathbf{a} + 6\mathbf{b} = 3(-\mathbf{a} + 2\mathbf{b}), \text{ so } \overrightarrow{MC} = 3\overrightarrow{MN}$$

So MN and MC are parallel, and they share the point M .

So M , N and C lie on a straight line.

Tip: Factorise to show the common part, e.g. $3(-\mathbf{a} + 2\mathbf{b})$. Then say “parallel and a common point M ”.

Marks: 1 mark for $\overrightarrow{AN}$ (or $\overrightarrow{ON}$). 1 mark for $\overrightarrow{MN} = -\mathbf{a} + 2\mathbf{b}$. 1 mark for $\overrightarrow{MC} = -3\mathbf{a} + 6\mathbf{b}$. 1 mark for $\overrightarrow{MC} = 3\overrightarrow{MN}$ with the conclusion (common point).