

Turning Points by Completing the Square

Solutions

For AQA GCSE Mathematics (8300) • Higher tier

Each question is shown in grey, followed by the working, one step per line. The final answer is in a box. *Marks* shows how the marks are usually given in the exam, and **Tip** points out a common mistake.

1 (4 marks)

- (a) Write $x^2 - 6x + 11$ in the form $(x - a)^2 + b$

Halve the coefficient of x : $-6 \div 2 = -3$

$$(x - 3)^2 = x^2 - 6x + 9$$

$$x^2 - 6x + 11 = (x - 3)^2 - 9 + 11$$

$$= (x - 3)^2 + 2$$

Answer: $(x - 3)^2 + 2$

Marks: 1 mark for $(x - 3)^2$. 1 mark for the correct answer.

- (b) Hence write down the coordinates of the turning point of $y = x^2 - 6x + 11$

$$y = (x - 3)^2 + 2$$

$(x - 3)^2$ is never negative. It is smallest (equal to 0) when $x = 3$.

$$\text{Then } y = 0 + 2 = 2$$

Answer: (3, 2)

- (c) Is the turning point a maximum or a minimum? Tick a box.

The x^2 term is positive, so the curve is U-shaped.

So the turning point is the lowest point: a minimum.

Answer: Minimum

2 (1 mark)

Circle the coordinates of the turning point of $y = (x + 2)^2 - 5$

Options: (2, -5), (-2, -5), (-2, 5), (2, 5)

$(x + 2)^2$ is smallest (equal to 0) when $x + 2 = 0$, so when $x = -2$.

Then $y = 0 - 5 = -5$

Answer: (-2, -5)

Tip: For $y = (x + p)^2 + q$ the turning point is $(-p, q)$. The x -coordinate has the **opposite** sign to the number in the bracket, but the y -coordinate keeps its sign.

3 (3 marks)

By completing the square, find the coordinates of the turning point of $y = -x^2 + 2x + 3$
State whether it is a maximum or a minimum.

Take out -1 from the x^2 and x terms: $-(x^2 - 2x) + 3$

Complete the square: $x^2 - 2x = (x - 1)^2 - 1$

So $y = -[(x - 1)^2 - 1] + 3$

$= -(x - 1)^2 + 1 + 3$

$= 4 - (x - 1)^2$

$(x - 1)^2$ is never negative, so y is never more than 4.

The greatest value is $y = 4$, when $x = 1$.

The x^2 term is negative, so the curve is n-shaped and this is a maximum.

Answer: (1, 4), maximum

Tip: When x^2 has a minus sign, take out -1 first. Then take care with the signs: $-(-1) = +1$.

Marks: 1 mark for $4 - (x - 1)^2$. 1 mark for (1, 4). 1 mark for maximum.

4 (3 marks)

Find the coordinates of the turning point of $y = 3x^2 - 12x + 5$

Take out the factor 3 from the x^2 and x terms: $3(x^2 - 4x) + 5$

Complete the square: $x^2 - 4x = (x - 2)^2 - 4$

So $y = 3[(x - 2)^2 - 4] + 5$

$= 3(x - 2)^2 - 12 + 5$

$= 3(x - 2)^2 - 7$

$3(x - 2)^2$ is smallest (equal to 0) when $x = 2$.

Then $y = -7$

Answer: (2, -7)

Tip: Multiply the -4 by the 3 outside the bracket: $3 \times (-4) = -12$. A common mistake is to write $3(x - 2)^2 - 4 + 5$.

Marks: 1 mark for $3(x^2 - 4x)$. 1 mark for $3(x - 2)^2 - 7$. 1 mark for the correct answer.

5 (3 marks)

The turning point of $y = x^2 + bx + c$ is at (4, -3)

Work out the values of b and c .

A turning point at (4, -3) means the completed square form is $y = (x - 4)^2 - 3$.

Expand: $(x - 4)^2 = x^2 - 8x + 16$

So $y = x^2 - 8x + 16 - 3 = x^2 - 8x + 13$

Compare with $x^2 + bx + c$:

$$b = -8 \text{ and } c = 13$$

Answer: $b = -8$, $c = 13$

Marks: 1 mark for $(x - 4)^2 - 3$. 1 mark for $b = -8$. 1 mark for $c = 13$.

6 (3 marks)

Tom says,

“The turning point of the graph of $y = x^2 + 14x + 40$ is at $(7, -9)$.”

Tom is wrong.

Complete the square to find the correct turning point, and explain Tom’s mistake.

Halve 14 to get 7.

$$(x + 7)^2 = x^2 + 14x + 49$$

$$x^2 + 14x + 40 = (x + 7)^2 - 49 + 40 = (x + 7)^2 - 9$$

$(x + 7)^2$ is smallest (equal to 0) when $x + 7 = 0$, so when $x = -7$.

So the turning point is $(-7, -9)$.

Tom has the wrong sign for the x -coordinate: $(x + 7)^2$ gives $x = -7$, not $x = 7$.

Answer: $(-7, -9)$. Tom used $x = 7$, but $(x + 7)^2 = 0$ when $x = -7$.

Tip: The turning point of $(x + a)^2 + b$ is at $(-a, b)$. The sign of a changes, but the sign of b does not.

Marks: 1 mark for $(x + 7)^2 - 9$. 1 mark for $(-7, -9)$. 1 mark for explaining the sign error in the x -coordinate.

7 (4 marks)

- (a) The turning point of $y = x^2 - 8x + c$ lies on the x -axis.
Work out the value of c .

Complete the square, keeping c as a letter:

$$x^2 - 8x + c = (x - 4)^2 - 16 + c$$

So the turning point is $(4, c - 16)$.

On the x -axis, $y = 0$, so $c - 16 = 0$.

$$c = 16$$

Answer: $c = 16$

Marks: 1 mark for $(x - 4)^2 - 16 + c$. 1 mark for the correct answer.

- (b) The turning point of $y = x^2 + 2x + d$ lies on the line $y = 2x$
Work out the value of d .

$$x^2 + 2x + d = (x + 1)^2 - 1 + d$$

So the turning point is $(-1, d - 1)$.

On the line $y = 2x$: when $x = -1$, $y = 2 \times (-1) = -2$.

$$\text{So } d - 1 = -2$$

$$d = -1$$

Answer: $d = -1$

Tip: You can complete the square with a letter in it. Just carry the letter along, then use the extra fact about the turning point.

Marks: 1 mark for the turning point $(-1, d - 1)$. 1 mark for the correct answer.

8 (3 marks)

The graph of $y = (x - p)^2 + q$ passes through the points $(-5, 21)$ and $(3, 21)$.
Work out the coordinates of the turning point of the graph.

The graph is symmetrical about a vertical line through the turning point.

$(-5, 21)$ and $(3, 21)$ have the same y -coordinate, so the line of symmetry is halfway between them.

$$x = \frac{-5 + 3}{2} = -1, \text{ so } p = -1.$$

The equation is $y = (x + 1)^2 + q$

Substitute $(3, 21)$: $21 = (3 + 1)^2 + q = 16 + q$

$$q = 5$$

The turning point is $(p, q) = (-1, 5)$.

Answer: $(-1, 5)$

Tip: Two points with the same y -value are the same distance either side of the turning point. Use the midpoint of their x -values.

Marks: 1 mark for $x = -1$ (the line of symmetry). 1 mark for substituting a point to get $q = 5$. 1 mark for $(-1, 5)$.
