

Iteration

Solutions

For AQA GCSE Mathematics (8300) • Higher tier

Each question is shown in grey, followed by the working, one step per line. The final answer is in a box. *Marks* shows how the marks are usually given in the exam, and **Tip** points out a common mistake.

1 (2 marks)

A sequence of numbers is formed by the iterative process

$$u_{n+1} = 2u_n - 3 \quad u_1 = 5$$

Work out the values of u_2 and u_3

The formula says: to get the next term, double the term you have and subtract 3.

$$u_2 = 2 \times 5 - 3 = 7$$

$$u_3 = 2 \times 7 - 3 = 11$$

Answer: $u_2 = 7$, $u_3 = 11$

Tip: Each new term comes from the term **before** it. Do not substitute $n = 1, 2, 3, \dots$ into the formula.

Marks: 1 mark for $u_2 = 7$. 1 mark for $u_3 = 11$ (or a correct follow-through from your u_2).

2 (3 marks)

Use the iteration formula $x_{n+1} = 3 - \frac{1}{x_n}$ with $x_1 = 2$ to work out x_2 , x_3 and x_4

Give x_4 to 4 decimal places.

Put each answer back into the formula to get the next one.

$$x_2 = 3 - \frac{1}{2} = 2.5$$

$$x_3 = 3 - \frac{1}{2.5} = 2.6$$

$$x_4 = 3 - \frac{1}{2.6} = 2.615384 \dots$$

Round to 4 decimal places.

Answer: 2.6154

Tip: The starting value here is x_1 , so the first value you work out is x_2 . Students who treated the start as x_0 gave the wrong term.

Marks: 1 mark for $x_2 = 2.5$. 1 mark for $x_3 = 2.6$. 1 mark for 2.6154.

3 (3 marks)

Here is a formula for an iterative process.

$$u_{n+1} = \frac{30}{u_n} + 2$$

$$u_2 = 8$$

Work out the values of u_1 and u_3

To find u_1 , work backwards.

Using the formula with $n = 1$: $u_2 = \frac{30}{u_1} + 2$

$$\text{So } \frac{30}{u_1} + 2 = 8$$

$$\text{Subtract 2: } \frac{30}{u_1} = 6$$

$$\text{So } u_1 = 30 \div 6 = 5$$

To find u_3 , work forwards from $u_2 = 8$:

$$u_3 = \frac{30}{8} + 2 = 3.75 + 2 = 5.75$$

Answer: $u_1 = 5$, $u_3 = 5.75$

Tip: Many students only found u_3 . To find the term **before** the one you know, put it into the formula as the output and solve an equation.

Marks: 1 mark for $\frac{30}{u_1} + 2 = 8$. 1 mark for $u_1 = 5$. 1 mark for $u_3 = 5.75$.

4 (1 mark)

Which equation can be rearranged to give $x = \sqrt{5x - 2}$?

Circle your answer.

Options: $x^2 - 5x + 2 = 0$, $x^2 + 5x - 2 = 0$, $x^2 - 5x - 2 = 0$, $x^2 = 5x + 2$

Square both sides of $x = \sqrt{5x - 2}$:

$$x^2 = 5x - 2$$

Subtract $5x$ and add 2: $x^2 - 5x + 2 = 0$

Answer: $x^2 - 5x + 2 = 0$

5 (6 marks)

(a) Show that the equation $x^3 - 6x + 2 = 0$ has a solution between $x = 0$ and $x = 1$

Substitute $x = 0$ and $x = 1$ into $x^3 - 6x + 2$.

When $x = 0$: $0 - 0 + 2 = 2$ (positive)

When $x = 1$: $1 - 6 + 2 = -3$ (negative)

The value changes sign, so there is a solution between 0 and 1.

Answer: 2 and -3 : there is a change of sign, so there is a solution between 0 and 1.

Tip: You must **say** that there is a change of sign. Just working out 2 and -3 does not get full marks.

Marks: 1 mark for working out both values. 1 mark for saying there is a change of sign, so there is a solution.

(b) Show that $x^3 - 6x + 2 = 0$ can be rearranged to give $x = \frac{x^3 + 2}{6}$

Start with $x^3 - 6x + 2 = 0$

Add $6x$ to both sides: $x^3 + 2 = 6x$

Divide both sides by 6: $\frac{x^3 + 2}{6} = x$, so $x = \frac{x^3 + 2}{6}$

Answer: $x = \frac{x^3 + 2}{6}$

- (c) Starting with $x_1 = 0$, use the iteration formula $x_{n+1} = \frac{x_n^3 + 2}{6}$ three times to find an estimate for the solution.
Give your answer to 4 decimal places.

$$x_2 = \frac{0^3 + 2}{6} = 0.33333 \dots$$

$$x_3 = \frac{0.33333 \dots^3 + 2}{6} = 0.339506 \dots$$

$$x_4 = \frac{0.339506 \dots^3 + 2}{6} = 0.339855 \dots$$

Round x_4 to 4 decimal places.

Answer: 0.3399

Tip: Use the full calculator value each time (the ANS key helps). Rounding x_2 to 0.3 gives a wrong x_3 .

Marks: 1 mark for $x_2 = 0.333 \dots$ 1 mark for $x_3 = 0.3395 \dots$ 1 mark for 0.3399.

6 (3 marks)

Sanjay has £3000 in a savings account.

At the end of each year, the bank adds interest and then Sanjay takes out £400.

The amount in the account at the end of n years is S_n pounds, where

$$S_{n+1} = 1.04 \times S_n - 400 \quad S_0 = 3000$$

At the end of which year will the amount in the account first be less than £2000?

You must show your working.

$S_0 = 3000$ is the amount at the start. Work out one year at a time.

$$S_1 = 1.04 \times 3000 - 400 = 3120 - 400 = 2720$$

$$S_2 = 1.04 \times 2720 - 400 = 2828.80 - 400 = 2428.80$$

$$S_3 = 1.04 \times 2428.80 - 400 = 2525.952 - 400 = 2125.95 \text{ (still more than £2000)}$$

$$S_4 = 1.04 \times 2125.952 - 400 = 2210.99 \dots - 400 = 1810.99 \text{ (less than £2000)}$$

So the amount first falls below £2000 at the end of year 4.

Answer: Year 4 (£1810.99)

Tip: Multiply by 1.04 first, then subtract 400 — not the other way round. Show each value so the examiner can see when it drops below £2000.

Marks: 1 mark for $S_1 = 2720$. 1 mark for S_2 and S_3 correct (or follow-through). 1 mark for year 4 with S_3 and S_4 seen.

7 (3 marks)

A rectangle has width w cm and length $(w + 3)$ cm.

The area of the rectangle is 50 cm^2

An approximate value of w can be found using the iterative formula

$$w_{n+1} = \sqrt{50 - 3w_n}$$

The starting value is $w_1 = 5$

(a) Work out the values of w_2 and w_3

$$w_2 = \sqrt{50 - 3 \times 5} = \sqrt{35} = 5.916 \dots$$

$$w_3 = \sqrt{50 - 3 \times 5.916 \dots} = \sqrt{32.25 \dots} = 5.679 \dots$$

Answer: $w_2 = 5.916 \dots$, $w_3 = 5.679 \dots$

Tip: Use the full value of w_2 (not a rounded value) when you work out w_3 .

Marks: 1 mark for w_2 . 1 mark for w_3 (or a correct follow-through from your w_2).

(b) Keep using the iterative formula until two values in a row round to the same number to 1 decimal place.

Write down the width of the rectangle to 1 decimal place.

$$w_3 = 5.679 \dots, \text{ which rounds to } 5.7$$

$$w_4 = \sqrt{50 - 3 \times 5.679 \dots} = 5.741 \dots, \text{ which also rounds to } 5.7$$

Two values in a row round to 5.7, so the width is 5.7 cm to 1 d.p.

(Further values, 5.725 ..., 5.729 ..., also round to 5.7.)

Answer: 5.7 cm

Tip: Stop when the values agree to the accuracy asked for, then give your answer to that accuracy.

8 (3 marks)

$$x_{n+1} = 6 - \frac{2}{x_n}$$

Use $x_1 = 1$ to work out an approximate solution to $x = 6 - \frac{2}{x}$
Give your answer to 4 significant figures.

$$x_2 = 6 - \frac{2}{1} = 4$$

$$x_3 = 6 - \frac{2}{4} = 5.5$$

$$x_4 = 5.6363 \dots$$

$$x_5 = 5.6451 \dots$$

$$x_6 = 5.64571 \dots$$

$$x_7 = 5.64575 \dots$$

x_6 and x_7 both round to 5.646 (4 significant figures), so stop here.

Answer: 5.646

Marks: 1 mark for $x_2 = 4$. 1 mark for continuing to at least x_5 . 1 mark for 5.646.
