

Inverse Functions

Solutions

For AQA GCSE Mathematics (8300) • Higher tier

Each question is shown in grey, followed by the working, one step per line. The final answer is in a box. *Marks* shows how the marks are usually given in the exam, and **Tip** points out a common mistake.

1 (6 marks)

Find the inverse function of each of these.

(a) $f(x) = 5x - 3$

Write $y = 5x - 3$ and rearrange to make x the subject.

Add 3 to both sides: $y + 3 = 5x$

Divide both sides by 5: $x = \frac{y + 3}{5}$

Now write the inverse in terms of x .

Answer: $f^{-1}(x) = \frac{x + 3}{5}$

Tip: $f^{-1}(x)$ means the **inverse** function. It is not $\frac{1}{f(x)}$ — the -1 is not a power here. This is the most common mistake in exams.

Marks: 1 mark for a correct first step, e.g. $y + 3 = 5x$. 1 mark for the correct answer.

(b) $g(x) = \frac{x}{4} + 7$

Write $y = \frac{x}{4} + 7$

Subtract 7 from both sides: $y - 7 = \frac{x}{4}$

Multiply both sides by 4: $x = 4(y - 7)$

Now write the inverse in terms of x .

Answer: $g^{-1}(x) = 4(x - 7)$ or $4x - 28$

Marks: 1 mark for a correct first step, e.g. $y - 7 = \frac{x}{4}$. 1 mark for the correct answer.

(c) $k(x) = \sqrt[3]{x} + 2$

Write $y = \sqrt[3]{x} + 2$

Subtract 2 from both sides: $y - 2 = \sqrt[3]{x}$

Cube both sides: $x = (y - 2)^3$

Now write the inverse in terms of x .

Answer: $k^{-1}(x) = (x - 2)^3$

Marks: 1 mark for $y - 2 = \sqrt[3]{x}$. 1 mark for the correct answer.

2 (2 marks)

$$f(x) = \sqrt{x + 5}$$

Work out $f^{-1}(4)$

$f^{-1}(4)$ is the input to f that gives an output of 4.

So solve $\sqrt{x + 5} = 4$

Square both sides: $x + 5 = 16$

Subtract 5: $x = 11$

(Or find $f^{-1}(x) = x^2 - 5$, then $f^{-1}(4) = 4^2 - 5 = 11$.)

Check: $f(11) = \sqrt{16} = 4$ ✓

Answer: 11

Tip: Do not work out $f(4)$. You are looking for the value that f turns **into** 4.

Marks: 1 mark for $\sqrt{x + 5} = 4$ or $f^{-1}(x) = x^2 - 5$. 1 mark for 11.

3 (2 marks)

$$f(x) = x^2 \text{ for } x \geq 0$$

The graph of $y = f(x)$ for $0 \leq x \leq 2$ is drawn on the grid.

On the same grid, draw the graph of $y = f^{-1}(x)$ for $0 \leq x \leq 4$

The opposite of squaring (for $x \geq 0$) is taking the square root.

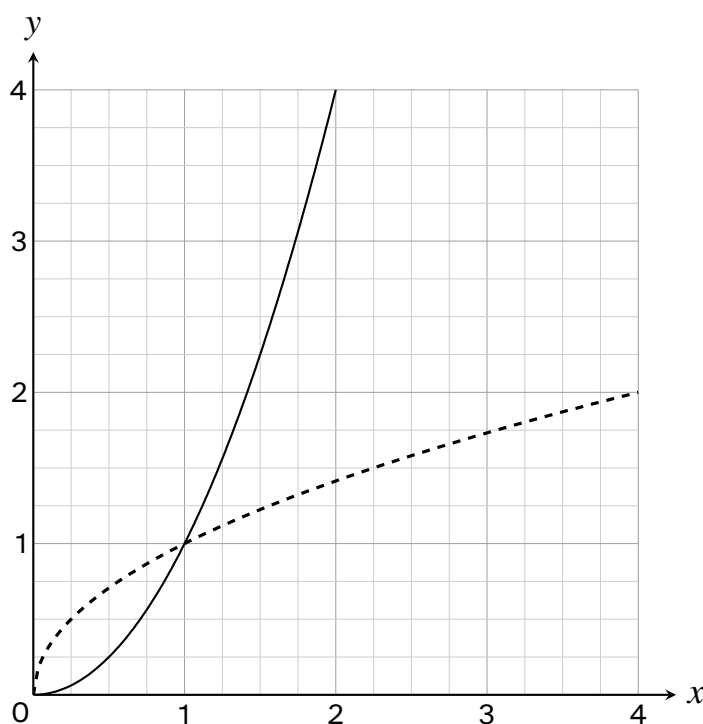
$$\text{So } f^{-1}(x) = \sqrt{x}$$

Work out some points on $y = \sqrt{x}$:

(0, 0), (1, 1), (2.25, 1.5), (4, 2)

(These are the points of $y = x^2$ with the coordinates swapped.)

Plot the points and join them with a smooth curve.



Answer: The curve $y = \sqrt{x}$ from (0, 0) to (4, 2) (dashed)

Tip: The graph of an inverse function is the reflection of the original graph in the line $y = x$: swap the coordinates of each point. The inverse is not $y = \frac{1}{x^2}$.

Marks: 1 mark for $f^{-1}(x) = \sqrt{x}$ or at least 2 correct points plotted, e.g. (1, 1) and (4, 2). 1 mark for a smooth curve from (0, 0) to (4, 2).

4 (4 marks)

$$f(x) = \frac{3x+2}{x-3}$$

Show that $f^{-1}(x) = f(x)$

$$\text{Write } y = \frac{3x+2}{x-3}$$

Multiply both sides by $(x-3)$: $y(x-3) = 3x+2$

Expand the bracket: $xy - 3y = 3x + 2$

Get the x terms on one side and everything else on the other:

$$xy - 3x = 3y + 2$$

Factorise out x : $x(y-3) = 3y+2$

$$\text{Divide by } (y-3): x = \frac{3y+2}{y-3}$$

So $f^{-1}(x) = \frac{3x+2}{x-3}$, which is the same as $f(x)$.

Answer: $f^{-1}(x) = \frac{3x+2}{x-3} = f(x)$

Tip: To get x on its own when it appears twice, collect the x terms on one side and factorise. Finish by writing a conclusion.

Marks: 1 mark for multiplying out, $xy - 3y = 3x + 2$. 1 mark for collecting the x terms on one side. 1 mark for factorising, $x(y-3) = 3y+2$. 1 mark for $f^{-1}(x) = \frac{3x+2}{x-3}$ with a conclusion.

5 (3 marks)

$$f(x) = 3x + 2$$

Solve $f(x) = f^{-1}(x)$

First find the inverse.

$$y = 3x + 2, \text{ so } y - 2 = 3x$$

$$\text{So } f^{-1}(x) = \frac{x-2}{3}$$

$$\text{Now solve } 3x + 2 = \frac{x-2}{3}$$

Multiply both sides by 3: $9x + 6 = x - 2$

Subtract x and 6 from both sides: $8x = -8$

Divide by 8: $x = -1$

Check: $f(-1) = -1$ and $f^{-1}(-1) = \frac{-3}{3} = -1$ ✓

Answer: $x = -1$

Marks: 1 mark for $f^{-1}(x) = \frac{x-2}{3}$. 1 mark for an equation without fractions, e.g. $9x + 6 = x - 2$. 1 mark for $x = -1$.

6 (4 marks)

$$f(x) = \frac{x+4}{3} \quad \text{and} \quad g(x) = x^2 - 2$$

(a) Work out $gf(5)$

$gf(5)$ means: put 5 into f first, then put the answer into g .

$$f(5) = \frac{5+4}{3} = 3$$

$$g(3) = 3^2 - 2 = 7$$

Answer: 7

Tip: Work from the right: $gf(5)$ uses f first. Working out $fg(5)$ instead gives 9.

Marks: 1 mark for $f(5) = 3$. 1 mark for 7.

(b) Jo says,

$$'f(5) = 3, \text{ so } f^{-1}(5) = \frac{1}{3},'$$

Is Jo correct?

You must show your working.

$f^{-1}(5)$ is the value of x that makes $f(x) = 5$.

$$\text{So solve } \frac{x+4}{3} = 5$$

Multiply by 3: $x + 4 = 15$

Subtract 4: $x = 11$

(Or find $f^{-1}(x) = 3x - 4$ and substitute $x = 5$.)

So $f^{-1}(5) = 11$, not $\frac{1}{3}$. Jo is not correct.

Answer: No. $f^{-1}(5) = 11$

Tip: $f^{-1}(5)$ is not $\frac{1}{f(5)}$. The -1 means the inverse function, not a reciprocal. This is exactly Jo's mistake.

Marks: 1 mark for $\frac{x+4}{3} = 5$ or $f^{-1}(x) = 3x - 4$. 1 mark for 11 and saying Jo is wrong.

7 (4 marks)

$f(x) = 2x + a$ and $f^{-1}(x) = bx + 3$ where a and b are constants.
Work out the value of a and the value of b .

Find the inverse of f in terms of a .

$$y = 2x + a, \text{ so } y - a = 2x$$

$$\text{So } f^{-1}(x) = \frac{x - a}{2} = \frac{1}{2}x - \frac{a}{2}$$

Compare with $bx + 3$:

$$\text{The coefficients of } x \text{ match: } b = \frac{1}{2}$$

$$\text{The constant terms match: } -\frac{a}{2} = 3$$

$$\text{So } a = -6$$

$$\text{Check: } f(x) = 2x - 6 \text{ gives } f^{-1}(x) = \frac{x+6}{2} = \frac{1}{2}x + 3 \quad \checkmark$$

Answer: $a = -6$, $b = \frac{1}{2}$

Tip: Split $\frac{x-a}{2}$ into $\frac{1}{2}x - \frac{a}{2}$ so that you can match the x terms and the numbers separately.

Marks: 1 mark for $f^{-1}(x) = \frac{x-a}{2}$. 1 mark for $b = \frac{1}{2}$. 1 mark for $-\frac{a}{2} = 3$. 1 mark for $a = -6$.

8 (5 marks)

$$f(x) = x^2 \quad g(x) = 2x - 1$$

Solve $gf(x) = g^{-1}(x)$

Find $gf(x)$ by putting $f(x) = x^2$ into g :

$$gf(x) = g(x^2) = 2x^2 - 1$$

Find the inverse of g :

$$y = 2x - 1, \text{ so } y + 1 = 2x, \text{ so } g^{-1}(x) = \frac{x+1}{2}$$

$$\text{Now solve } 2x^2 - 1 = \frac{x+1}{2}$$

$$\text{Multiply both sides by 2: } 4x^2 - 2 = x + 1$$

$$\text{Rearrange so one side is 0: } 4x^2 - x - 3 = 0$$

$$\text{Factorise: } (4x + 3)(x - 1) = 0$$

$$\text{So } 4x + 3 = 0 \text{ or } x - 1 = 0$$

$$\textbf{Answer: } x = 1 \text{ or } x = -\frac{3}{4}$$

Tip: $gf(x)$ means put $f(x)$ into g , so it is $2(x^2) - 1$, not $(2x - 1)^2$. It does not mean $g(x) \times f(x)$.

Marks: 1 mark for $gf(x) = 2x^2 - 1$. 1 mark for $g^{-1}(x) = \frac{x+1}{2}$. 1 mark for a quadratic equal to 0. 1 mark for factorising or using the formula. 1 mark for both answers.