

Completing the Square

Solutions

For AQA GCSE Mathematics (8300) • Higher tier

Each question is shown in grey, followed by the working, one step per line. The final answer is in a box. *Marks* shows how the marks are usually given in the exam, and **Tip** points out a common mistake.

1 (2 marks)

- (a) Write $x^2 + 10x$ in the form $(x + a)^2 - b$

Halve the coefficient of x : $10 \div 2 = 5$

$$(x + 5)^2 = x^2 + 10x + 25$$

This has an extra 25, so subtract 25:

$$x^2 + 10x = (x + 5)^2 - 25$$

Answer: $(x + 5)^2 - 25$

- (b) Write $x^2 - 14x + 9$ in the form $(x - a)^2 - b$

Halve the coefficient of x : $-14 \div 2 = -7$

$$(x - 7)^2 = x^2 - 14x + 49$$

$$\text{So } x^2 - 14x = (x - 7)^2 - 49$$

$$\text{Now add the 9: } x^2 - 14x + 9 = (x - 7)^2 - 49 + 9$$

$$= (x - 7)^2 - 40$$

Answer: $(x - 7)^2 - 40$

Tip: The sign in the bracket matches the sign of the x term: $-14x$ gives $(x - 7)^2$, not $(x + 7)^2$.

2 (6 marks)

Write each expression in the form $(x + a)^2 + b$

(a) $x^2 + 6x + 2$

Halve 6 to get 3.

$$(x + 3)^2 = x^2 + 6x + 9, \text{ so } x^2 + 6x = (x + 3)^2 - 9$$

$$x^2 + 6x + 2 = (x + 3)^2 - 9 + 2$$

$$= (x + 3)^2 - 7$$

Answer: $(x + 3)^2 - 7$

Tip: Do not forget to subtract the $3^2 = 9$. Leaving the answer as $(x + 3)^2 + 2$ is a very common mistake.

Marks: 1 mark for $(x + 3)^2$. 1 mark for the correct answer.

(b) $x^2 - 10x + 30$

Halve -10 to get -5 .

$$(x - 5)^2 = x^2 - 10x + 25, \text{ so } x^2 - 10x = (x - 5)^2 - 25$$

$$x^2 - 10x + 30 = (x - 5)^2 - 25 + 30$$

$$= (x - 5)^2 + 5$$

Answer: $(x - 5)^2 + 5$

Marks: 1 mark for $(x - 5)^2$. 1 mark for the correct answer.

(c) $x^2 + 5x + 1$

Halve 5 to get $\frac{5}{2}$.

$$\left(x + \frac{5}{2}\right)^2 = x^2 + 5x + \frac{25}{4}$$

$$\text{So } x^2 + 5x + 1 = \left(x + \frac{5}{2}\right)^2 - \frac{25}{4} + 1$$

$$-\frac{25}{4} + \frac{4}{4} = -\frac{21}{4}$$

Answer: $\left(x + \frac{5}{2}\right)^2 - \frac{21}{4}$

Marks: 1 mark for $\left(x + \frac{5}{2}\right)^2$. 1 mark for the correct answer.

3 (3 marks)

Solve $x^2 - 6x + 2 = 0$ by completing the square.
Give your answers in the form $a \pm \sqrt{b}$

Complete the square: halve -6 to get -3 .

$$x^2 - 6x + 2 = (x - 3)^2 - 9 + 2 = (x - 3)^2 - 7$$

$$\text{So } (x - 3)^2 - 7 = 0$$

$$\text{Add 7 to both sides: } (x - 3)^2 = 7$$

$$\text{Square root both sides (remember } \pm \text{): } x - 3 = \pm\sqrt{7}$$

$$\text{Add 3: } x = 3 \pm \sqrt{7}$$

Answer: $x = 3 + \sqrt{7}$ or $x = 3 - \sqrt{7}$

Tip: When you square root, write $\pm$. A positive number has two square roots, so there are two answers.

Marks: 1 mark for $(x - 3)^2 - 7$. 1 mark for $x - 3 = \pm\sqrt{7}$. 1 mark for the correct answer.

4 (3 marks)

(a) Write $x^2 - 4x + 7$ in the form $(x - a)^2 + b$

Halve -4 to get -2 .

$$(x - 2)^2 = x^2 - 4x + 4$$

$$x^2 - 4x + 7 = (x - 2)^2 - 4 + 7$$

$$= (x - 2)^2 + 3$$

Answer: $(x - 2)^2 + 3$

Marks: 1 mark for $(x - 2)^2$. 1 mark for the correct answer.

- (b) Hence explain why $x^2 - 4x + 7 > 0$ for all values of x .

$$x^2 - 4x + 7 = (x - 2)^2 + 3$$

A square is never negative, so $(x - 2)^2 \geq 0$.

$$\text{So } (x - 2)^2 + 3 \geq 3$$

3 is positive, so $x^2 - 4x + 7 > 0$ for all x .

Answer: A square is never negative, so the expression is always at least 3

5 (2 marks)

$$x^2 + px + q \equiv (x - 3)^2 + 2$$

Work out the values of p and q .

Expand the right side:

$$(x - 3)^2 + 2 = x^2 - 6x + 9 + 2 = x^2 - 6x + 11$$

Compare with $x^2 + px + q$:

The x terms give $p = -6$.

The number terms give $q = 11$.

Answer: $p = -6$, $q = 11$

Marks: 1 mark for $p = -6$. 1 mark for $q = 11$.

6 (3 marks)

Kai says,

“When x is between 5 and 7, the value of $x^2 - 12x + 39$ is negative.”

By completing the square, show that Kai is wrong.

Complete the square: halve -12 to get -6 .

$$x^2 - 12x + 39 = (x - 6)^2 - 36 + 39$$

$$= (x - 6)^2 + 3$$

$(x - 6)^2 \geq 0$ for all values of x , because a square is never negative.

So $(x - 6)^2 + 3 \geq 3$ for every value of x .

So the expression is never negative (its smallest value is 3, when $x = 6$).

So Kai is wrong.

Answer: $(x - 6)^2 + 3 \geq 3$, so the expression is never negative.

Tip: The question says *by completing the square*, so just substituting a value such as $x = 6$ will not get the marks. Use the completed square and say that $(x - 6)^2$ can never be negative.

Marks: 1 mark for $(x - 6)^2$. 1 mark for $(x - 6)^2 + 3$. 1 mark for a conclusion that uses $(x - 6)^2 \geq 0$.

7 (3 marks)

The n th term of a sequence is $n^2 - 24n + 155$

By completing the square, show that all the terms of the sequence have two or more digits.

Complete the square: halve -24 to get -12 .

$$n^2 - 24n + 155 = (n - 12)^2 - 144 + 155$$

$$= (n - 12)^2 + 11$$

$(n - 12)^2 \geq 0$, because a square is never negative.

So every term is at least 11.

The smallest term is 11, when $n = 12$.

11 has two digits, so every term has two or more digits.

Answer: $(n - 12)^2 + 11 \geq 11$

Tip: Finish by linking your answer to the question: the smallest possible term is 11, which has two digits.

Marks: 1 mark for $(n - 12)^2$. 1 mark for $(n - 12)^2 + 11$. 1 mark for the conclusion that every term is at least 11.
