

Algebraic Proof

Solutions

For AQA GCSE Mathematics (8300) • Higher tier

Each question is shown in grey, followed by the working, one step per line. The final answer is in a box. *Marks* shows how the marks are usually given in the exam, and **Tip** points out a common mistake.

1 (3 marks)

n is an integer.

Tick **one** box for each expression.

	Always even	Always odd	Either
$2n$	✓		
$2n + 1$		✓	
$n + 1$			✓
$2n + 4$	✓		
n^2			✓
$4n - 1$		✓	

$2n = 2 \times n$, a multiple of 2, so it is always even.

$2n + 1$ is one more than a multiple of 2, so it is always odd.

$n + 1$ is even when n is odd, and odd when n is even. So it could be either.

$2n + 4 = 2(n + 2)$, a multiple of 2, so it is always even.

n^2 : $2^2 = 4$ is even, but $3^2 = 9$ is odd. So it could be either.

$4n - 1 = 2(2n - 1) + 1$, one more than a multiple of 2, so it is always odd.

Answer: See table

Marks: 3 marks for all six correct. 2 marks for four or five correct. 1 mark for two or three correct.

2 (1 mark)

p is an even number and q is an odd number.

Which expression is **always** odd?

Circle your answer.

Options: pq , $p^2 + q$, $4q$, $p + q + 1$

$pq = \text{even} \times \text{odd}$, which is even.

$p^2 = \text{even} \times \text{even}$, which is even. So $p^2 + q = \text{even} + \text{odd}$, which is odd.

$4q = 4 \times \text{odd}$, which is even.

$p + q + 1 = \text{even} + \text{odd} + 1 = \text{odd} + 1$, which is even.

Answer: $p^2 + q$

3 (2 marks)

Prove that the sum of three consecutive integers is always a multiple of 3

Let the first integer be n .

The next two integers are $n + 1$ and $n + 2$.

Sum: $n + (n + 1) + (n + 2) = 3n + 3$

Factorise: $3n + 3 = 3(n + 1)$

$n + 1$ is an integer, so $3(n + 1)$ is a multiple of 3.

Answer: $3(n + 1)$

Tip: Checking examples, such as $1 + 2 + 3 = 6$, is not a proof. You must use algebra so that it works for every integer.

Marks: 1 mark for $n + (n + 1) + (n + 2)$. 1 mark for $3(n + 1)$ with a conclusion.

4 (3 marks)

Prove that the product of two odd numbers is always odd.

The two odd numbers can be different, so use two letters.

Let the odd numbers be $2m + 1$ and $2n + 1$, where m and n are integers.

Multiply: $(2m + 1)(2n + 1) = 4mn + 2m + 2n + 1$

Take out 2 from the first three terms:

$$= 2(2mn + m + n) + 1$$

$2(2mn + m + n)$ is even, so adding 1 makes the product odd.

Answer: $2(2mn + m + n) + 1$

Marks: 1 mark for $(2m + 1)(2n + 1)$. 1 mark for expanding correctly. 1 mark for $2(2mn + m + n) + 1$ with a conclusion.

5 (2 marks)

Sam says, " $n^2 + n$ is always odd."

Prove that Sam is wrong.

Factorise: $n^2 + n = n(n + 1)$

n and $n + 1$ are consecutive integers, so one of them is even.

An even number multiplied by any integer is even.

So $n^2 + n$ is always even, and Sam is wrong.

(One example is also enough to show Sam is wrong: $n = 1$ gives $1 + 1 = 2$, which is even.)

Answer: $n(n + 1)$ is even

Tip: If you factorise, you must explain **why** $n(n + 1)$ is even: one of two consecutive integers is always even.

Marks: 1 mark for $n(n + 1)$, or for a correct example such as $n = 1$ giving 2. 1 mark for a correct reason why it is even, or for stating that the example is even so Sam is wrong.

6 (3 marks)

Two even numbers have a difference of 2

Prove that when they are multiplied together, the answer is always a multiple of 8

Let the two even numbers be $2n$ and $2n + 2$, where n is an integer.

Product: $2n(2n + 2) = 4n^2 + 4n$

Factorise: $= 4n(n + 1)$

n and $n + 1$ are consecutive integers, so one of them is even.

So $n(n + 1)$ is even. Write $n(n + 1) = 2k$, where k is an integer.

Then the product is $4 \times 2k = 8k$, which is a multiple of 8.

Answer: $4n(n + 1) = 8k$

Tip: $4n(n + 1)$ only shows a multiple of 4. For the last mark you must explain why $n(n + 1)$ is even.

Marks: 1 mark for $2n(2n + 2)$. 1 mark for $4n(n + 1)$. 1 mark for explaining that $n(n + 1)$ is even, so the product is a multiple of 8.

7 (3 marks)

Keira thinks of an integer that is not zero.

She adds 2 to it and squares the result.

She subtracts 4

She then divides by the integer she first thought of.

Prove algebraically that her final answer is always 4 more than the integer she first thought of.

Let Keira's integer be n .

Add 2 and square: $(n + 2)^2 = n^2 + 4n + 4$

Subtract 4: $n^2 + 4n + 4 - 4 = n^2 + 4n$

Divide by n : $\frac{n^2 + 4n}{n} = \frac{n(n + 4)}{n} = n + 4$

$n + 4$ is 4 more than the integer she first thought of.

Answer: $n + 4$

Tip: Square the whole of $n + 2$: $(n + 2)^2$ is $n^2 + 4n + 4$, not $n^2 + 4$.

Marks: 1 mark for $(n + 2)^2 - 4$. 1 mark for $n^2 + 4n$. 1 mark for dividing by n to get $n + 4$.

8 (2 marks)

x is a square number.

Show that the next square number is $x + 2\sqrt{x} + 1$

Let $x = n^2$, where n is a positive integer.

Then $n = \sqrt{x}$.

The next square number is $(n + 1)^2 = (\sqrt{x} + 1)^2$.

Expand: $(\sqrt{x} + 1)(\sqrt{x} + 1) = x + \sqrt{x} + \sqrt{x} + 1$

$$= x + 2\sqrt{x} + 1$$

Answer: Shown

Tip: A numerical example (such as 9 and 16) does not show it is always true. You need the algebra $(\sqrt{x} + 1)^2$.

Marks: 1 mark for $(\sqrt{x} + 1)^2$. 1 mark for expanding it correctly.

9 (3 marks)

The n th triangular number is $\frac{n(n+1)}{2}$

Prove that 8 times any triangular number, plus 1, is always a square number.

8 times the n th triangular number, plus 1:

$$8 \times \frac{n(n+1)}{2} + 1$$

$$= 4n(n+1) + 1$$

$$= 4n^2 + 4n + 1$$

$$\text{Factorise: } = (2n+1)^2$$

$2n+1$ is an integer, so this is a square number.

Answer: $(2n+1)^2$

Marks: 1 mark for $8 \times \frac{n(n+1)}{2} + 1$. 1 mark for $4n^2 + 4n + 1$. 1 mark for $(2n+1)^2$.

10 (3 marks)

(a) Write $x^2 - 10x + 32$ in the form $(x-a)^2 + b$

Half of -10 is -5 , so start with $(x-5)^2$.

$$(x-5)^2 = x^2 - 10x + 25$$

$$\text{So } x^2 - 10x + 32 = (x-5)^2 - 25 + 32$$

$$= (x - 5)^2 + 7$$

Answer: $(x - 5)^2 + 7$

Tip: The number in the bracket is half the coefficient of x : -5 , not -10 .

Marks: 1 mark for $(x - 5)^2$. 1 mark for $(x - 5)^2 + 7$.

- (b) Hence prove that $x^2 - 10x + 32 \geq 7$ for all values of x .

A square is never negative, so $(x - 5)^2 \geq 0$.

So $(x - 5)^2 + 7 \geq 0 + 7 = 7$.

So $x^2 - 10x + 32 \geq 7$ for all values of x .

Answer: $(x - 5)^2 \geq 0$, so $(x - 5)^2 + 7 \geq 7$

Tip: You must say **why** the smallest value is 7: $(x - 5)^2$ can never be negative.

11 (4 marks)

n is an integer.

Prove that $(n + 1)^3 - (n - 1)^3$ is always 2 more than a multiple of 6

$$(n + 1)^2 = n^2 + 2n + 1, \text{ so } (n + 1)^3 = (n + 1)(n^2 + 2n + 1) = n^3 + 3n^2 + 3n + 1$$

$$(n - 1)^2 = n^2 - 2n + 1, \text{ so } (n - 1)^3 = (n - 1)(n^2 - 2n + 1) = n^3 - 3n^2 + 3n - 1$$

Subtract, keeping the brackets:

$$(n^3 + 3n^2 + 3n + 1) - (n^3 - 3n^2 + 3n - 1)$$

$$= 6n^2 + 2$$

$6n^2 = 6 \times n^2$ is a multiple of 6, so $6n^2 + 2$ is 2 more than a multiple of 6.

Answer: $6n^2 + 2$

Tip: $(n + 1)^3$ is not $n^3 + 1$. Expand it as $(n + 1)(n + 1)(n + 1)$, one pair of brackets at a time.

Marks: 1 mark for expanding $(n + 1)^3$. 1 mark for expanding $(n - 1)^3$. 1 mark for $6n^2 + 2$. 1 mark for the conclusion.
